CHAPTER ONE

FUNCTION OF SEVERAL VARIABLES

After completing these tutorials, students should be able to:

- Find domain and range of given functions
- Sketch level curve and its graph of given functions
- Sketch surface graph define by x, y, and z in 3 dimensions
- Sketch the level surface graph of given functions
- Verify the limit of given functions
- Show that limit does not exist for a given function
- Find the area R in which the given function is continuous
- Find the first order partial derivatives of the given functions
- Find the second-order partial derivative of the given functions
- Find the approximation values by using partial derivative method
- Find the partial derivative of the given functions by using the chain rule
- Find the local extremum of the given functions

Question 1

Find domain and range for f ,then state the value of f for the points given:

(a) $f(x, y) = xy/(x-2y); (2,3), (-1,4)$

Solution:

x and y is defined in condition $x-2y \neq 0$, such as $x \neq 2y$.

$D_f = \{(x, y): x, y \in \Re, x \neq 2y\}$, $R_f = \{f(x, y): f(x, y) \in \Re\}$.

$f(2,3) = \dfrac{(2)(3)}{2-2(3)} = \dfrac{6}{-4} = -\dfrac{3}{2}$, $f(-1,4) = \dfrac{(-1)(4)}{-1-2(4)} = \dfrac{-4}{-9} = \dfrac{4}{9}$

(b) $f(x, y, z) = \sqrt{25 - x^2 - y^2 - z^2}; (1,-2,2), (-3,0,2)$

Solution:

x, y and z is defined in condition $25 - x^2 - y^2 - z^2 \geq 0$, such as $x^2 + y^2 + z^2 \leq 25$.

$D_f = \{(x, y, z): x, y, z \in \Re, x^2 + y^2 + z^2 \leq 25\}$, $R_f = \{f(x, y, z): 0 \leq f(x, y, z) \leq 5\}$.

$f(1,-2,2) = \sqrt{25 - (1)^2 - (-2)^2 - (2)^2} = \sqrt{25-1-4-4} = \sqrt{16} = 4$,

$f(-3,0,2) = \sqrt{25 - (-3)^2 - (0)^2 - (2)^2} = \sqrt{25-9-0-4} = \sqrt{12} = 2\sqrt{3}$

(c) $f(x, y, z, w) = yzw + \sqrt{x-5}; (6,1,1,1), (9,2,1,1)$

Solution:

y, z and w can be any values while x is defined only for $x-5 \geq 0$, such as $x \geq -5$.

$D_f = \{(x, y, z): x, y, z \in \Re, x^2 + y^2 + z^2 \leq 25\}$, $R_f = \{f(x, y, z): 0 \leq f(x, y, z) \leq 5\}$.

$f(6,1,1,1) = (1)(1)(1) + \sqrt{6-5} = 1+1 = 2$, $f(9,2,1,1) = (2)(1)(1) + \sqrt{9-5} = 2+2 = 4$

Question 2

Sketch level curve for the following functions and then sketch its graph:

(a) $z=5\left(\sqrt{1-\frac{x^2}{16}-\frac{y^2}{9}}\right)$ for $k=0,\frac{5}{3},\frac{10}{3}$

Solution:

$$z=5\left(\sqrt{1-\frac{x^2}{16}-\frac{y^2}{9}}\right)=k$$

$$\sqrt{1-\frac{x^2}{16}-\frac{y^2}{9}}=\frac{k}{5}\to 1-\frac{x^2}{16}-\frac{y^2}{9}=\frac{k^2}{25}\to\frac{x^2}{16}+\frac{y^2}{9}=1-\frac{k^2}{25}$$

When k=0, $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

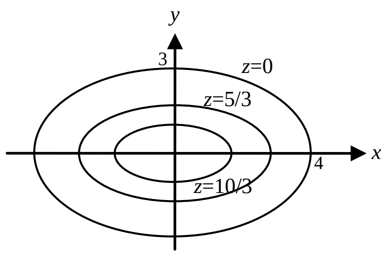
When k= $\frac{5}{3}$, $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1-\frac{5^2}{3^2}\cdot\frac{1}{5^2}=\frac{8}{9}$

$$\frac{x^2}{4^2}+\frac{y^2}{3^2}=\frac{8}{9}$$

$$\frac{x^2}{4^2}\cdot\frac{9}{8}+\frac{y^2}{3^2}\cdot\frac{9}{8}=1$$

$$\frac{x^2}{128/9}+\frac{y^2}{8}=1$$

$$\frac{x^2}{\left(\sqrt{128/9}\right)^2}+\frac{y^2}{\left(\sqrt{8}\right)^2}=1\to(3.78,2.83)$$



Level Curve

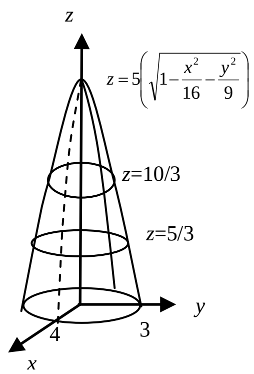
When k= $\frac{10}{3}$, $\frac{x^2}{4^2}+\frac{y^2}{3^2}=1-\frac{10^2}{3^2}\cdot\frac{1}{5^2}=\frac{5}{9}$

$$\frac{x^2}{4^2}+\frac{y^2}{3^2}=\frac{5}{9}$$

$$\frac{x^2}{4^2}\cdot\frac{9}{5}+\frac{y^2}{3^2}\cdot\frac{9}{5}=1$$

$$\frac{x^2}{80/9}+\frac{y^2}{5}=1$$

$$\frac{x^2}{\left(\sqrt{80/9}\right)^2}+\frac{y^2}{\left(\sqrt{5}\right)^2}=1\to(2.98,2.24)$$



(b) $z = \sqrt{x^2 + y^2}$ for $k = 2,3,4$

<u>Solution:</u>

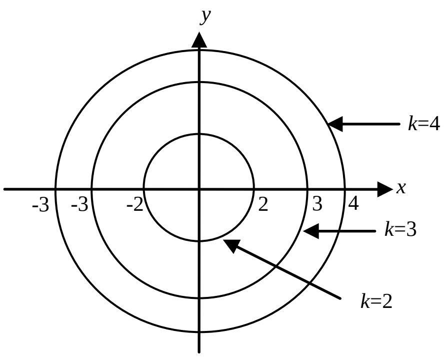
$z = \sqrt{x^2 + y^2} = k$

$x^2 + y^2 = k^2$

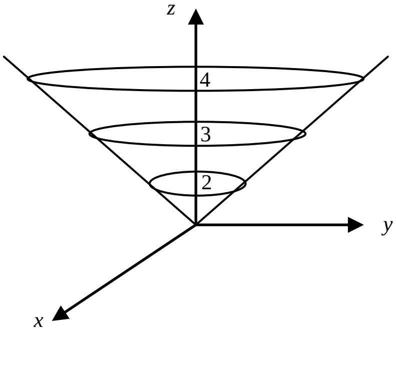
When k=2, $x^2 + y^2 = 4$
A circle with radius 2.

When k=3, $x^2 + y^2 = 9$
A circle with radius 3.

When k=4, $x^2 + y^2 = 16$
A circle with radius 4.



Level Curve



Question 3

Sketch surface graph define by x, y, and z in 3 dimensions

(a) $2x + 3y + 4z = 11$

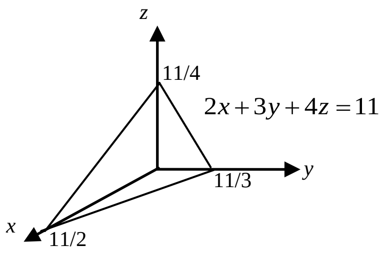
Solution:

$2x + 3y + 4z = 11$

When x=0, y=0 $4z = 11 \to z = 11/4$. $(0,0,11/4)$

When x=0, z=0 $3y = 11 \to y = 11/3$. $(0,11/3,0)$

When y=0, z=0 $2x = 11 \to x = 11/2$. $(11/2,0,0)$



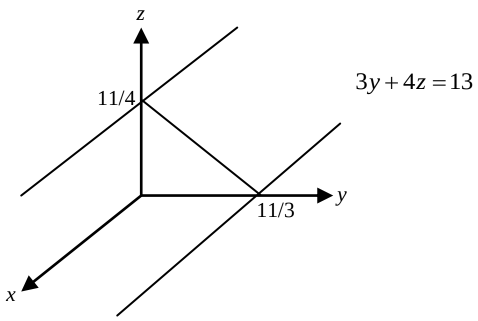
(b) $3y + 4z = 13$

Solution

$3y + 4z = 13$

When y=0 $4z = 13 \to z = 13/4$. $(x,0,13/4)$

When z=0 $3y = 13 \to y = 13/3$. $(x,13/3,0)$



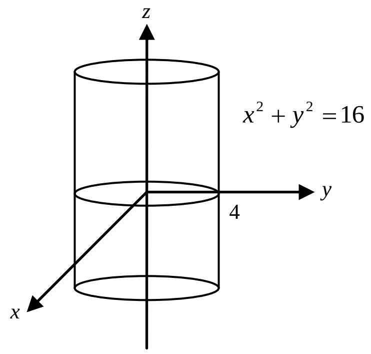
(c) $x^2 + y^2 = 16$

Solution:

$x^2 + y^2 = 16$.

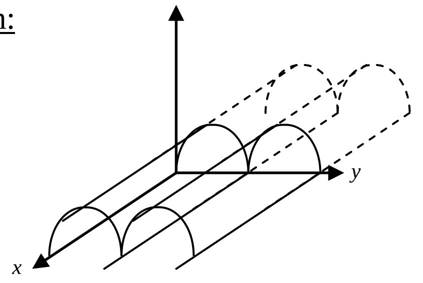
$\frac{x^2}{16} + \frac{y^2}{16} = 1$.

A cylinder with radius 4.



(d) $z = |\sin y|$

Solution:



Question 4

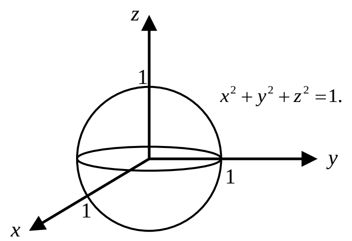
Sketch the level surface graph for the functions below when k =1:

(a) $f(x, y, z) = x^2 + y^2 + z^2$

Solution:

$f(x, y, z) = x^2 + y^2 + z^2 = k$

When k=1, $x^2 + y^2 + z^2 = 1$. A sphere with radius 1.

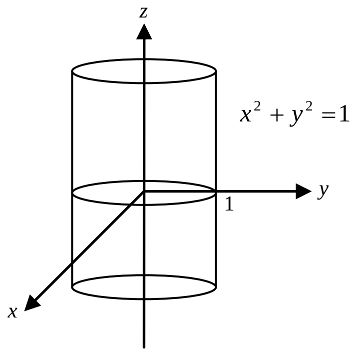


(b) $f(x, y, z) = x^2 + y^2$

Solution:

$f(x, y, z) = x^2 + y^2 = k$

When k=1, $x^2 + y^2 = 1$. A cylinder with radius 1.



Question 5

Verify the limit for the following functions:

(a) $\lim_{(x,y)\to(0,0)} \ln(1+x^2y^3)$

Solution:

$$\lim_{(x,y)\to(0,0)} \ln(1+x^2y^3) = \ln(1+0) = \ln 1 = 0$$

(b) $\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$

Solution:

$$\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y} = \frac{(-1)(2)^3}{-1+2} = \frac{-8}{1} = -8$$

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2-2}{3+xy}$

Solution:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-2}{3+xy} = \frac{0-2}{3+0} = -\frac{2}{3}$$

Question 6

Show that limit does not exist for the following functions:

(a) $\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+2y^2}$

Solution:

Consider the first line, $x=0$ and compute the limit as y approaches 0. Thus, we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2-y^2}{x^2+2y^2} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{2y^2} = \lim_{(x,y)\to(0,0)} \frac{-1}{2} = \frac{-1}{2}$$

Similarly, for the second line, $y=0$ and compute the limit as x approaches 0. Thus, we have

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + 2y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{x^2} = \lim_{(x,y)\to(0,0)} 2 = 2$$

Since the limits from two different directions are not the same, therefore $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$ does not exist.

(b) $\lim_{(x,y)\to(0,1)} \frac{x(y-1)}{x^2 + y^2 - 2y + 1}$

Solution:

Consider the first line, $y = x + 1$ and compute the limit as y approaches 1. Thus, we have

$$\lim_{(x,y)\to(0,1)} \frac{x(y-1)}{x^2 + y^2 - 2y + 1} = \lim_{(x,y)\to(0,1)} \frac{x(x+1-1)}{x^2 + (x+1)^2 - 2(x+1) + 1}$$

$$= \lim_{(x,y)\to(0,1)} \frac{x(x+1-1)}{x^2 + x^2 + 2x + 1 - 2x - 2 + 1} = \lim_{(x,y)\to(0,1)} \frac{x^2}{2x^2} = \frac{1}{2}$$

Similarly, for the second line, $y = -x + 1$ and compute the limit as x approaches 0. Thus, we have

$$\lim_{(x,y)\to(0,1)} \frac{x(y-1)}{x^2 + y^2 - 2y + 1} = \lim_{(x,y)\to(0,1)} \frac{x(-x+1-1)}{x^2 + (-x+1)^2 - 2(-x+1) + 1}$$

$$= \lim_{(x,y)\to(0,1)} \frac{x(-x+1-1)}{x^2 + x^2 - 2x + 1 + 2x - 2 + 1} = \lim_{(x,y)\to(0,1)} \frac{-x^2}{2x^2} = -\frac{1}{2}$$

Since the limits from two different directions are not the same, therefore $\lim_{(x,y)\to(0,1)} \frac{x(y-1)}{x^2 + y^2 - 2y + 1}$ does not exist.

Question 7

Find the area *R* in which the following function f is continuous:

(a) $f(x,y) = \ln(x + y - 1)$

Solution:

$x + y - 1 > 0 \Rightarrow x + y > 1.$

Area of $R = \{(x,y): x, y \in \Re, x + y > 1\}$

(b) $f(x,y) = \sqrt{x}e^{\sqrt{1-y}}$

Solution:

$x \geq 0,\ 1 - y \geq 0 \Rightarrow y \leq 1.$

Area of $R = \{(x,y): x, y \in \Re, x \geq 0, y \leq 1\}$

(c) $f(x,y) = xy\sin\left(\frac{1}{z}\right)$

Solution:

Area of $R = \{(x,y,z): x, y, z \in \Re, z \neq 0\}.$

Question 8

Find the first order partial derivatives, f_x and f_y of the given functions.

(a) $f(x,y)=x^2y+4y^3-5x+6$

Solution:

$$f_x=2xy-5,\quad f_y=x^2+12y^2$$

(b) $f(x,y)=\left(2x^3y^2-y\right)^2$

Solution:

$$f_x=12x^2y^2\left(2x^3y^2-y\right),\quad f_y=2\left(2x^3y^2-y\right)\left(4x^3y-1\right)$$

(c) $f(x,y)=\left(x+y\right)e^{(xy)^2}$

Solution:

$$f_x=\left(x+y\right)e^{(xy)^2}2\left(xy\right)y+e^{(xy)^2}=e^{(xy)^2}\left(2x^2y^2+2xy^3+1\right),$$

$$f_y=\left(x+y\right)e^{(xy)^2}2\left(xy\right)x+e^{(xy)^2}=e^{(xy)^2}\left(2x^3y+2x^2y^2+1\right)$$

(d) $f(x,y)=y^2\ln(x^2y)$

Solution:

$$f_x=y^2\cdot\frac{1}{x^2y}\cdot 2xy=\frac{2y^2}{x},\quad f_y=2y\ln\left(x^2y\right)+y^2\cdot\frac{1}{x^2y}x^2=2y\ln\left(x^2y\right)+y$$

Question 9

Find all the second-order partial derivative, $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ of $f(x,y)=\dfrac{e^{x+y}}{x^2+y^2}$ at point (0,1)

Solution:

$$f_x=\frac{\left(x+y\right)y-xy\left(1\right)}{\left(x+y\right)^2}=\frac{y^2}{\left(x+y\right)^2},\qquad f_y=\frac{\left(x+y\right)x-xy\left(1\right)}{\left(x+y\right)^2}=\frac{x^2}{\left(x+y\right)^2}$$

$$f_{xx} = \frac{(x+y)^2(0) - y^2 2(x+y)(1)}{(x+y)^4} = \frac{-2y^2}{(x+y)^3}, \qquad f_{xx}(0,1) = \frac{-2}{1} = -2$$

$$f_{xy} = \frac{(x+y)^2(2y) - y^2 2(x+y)(1)}{(x+y)^4} = \frac{(x+y)\left[(x+y)(2y) - 2y^2\right]}{(x+y)^4} = \frac{2yx}{(x+y)^3}, \quad f_{xy}(0,1) = \frac{2(1)(0)}{1} = 0$$

$$f_{yy} = \frac{(x+y)^2(0) - x^2 2(x+y)(1)}{(x+y)^4} = \frac{-2x^2}{(x+y)^3}, \qquad f_{yy}(0,1) = \frac{-2(0)}{1} = 0$$

$$f_{yx} = \frac{(x+y)^2(2x) - x^2 2(x+y)(1)}{(x+y)^4} = \frac{(x+y)\left[(x+y)(2x) - 2x^2\right]}{(x+y)^4} = \frac{2yx}{(x+y)^3}, \quad f_{yx}(0,1) = \frac{2(1)(0)}{1} = 0$$

Question 10

Let $z = f(x, y)$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

(a) $x^4 \sin(xy^3) - yz = 0$

Solution:

$$\frac{\partial}{\partial x}\left[x^4 \sin(xy^3)\right] - \frac{\partial}{\partial x}(yz) = \frac{\partial}{\partial x}(0)$$

$$4x^3 \sin(xy^3) + x^4 \cos(xy^3)\, y^3 - y\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{4x^3 \sin(xy^3) + x^4 y^3 \cos(xy^3)}{y}$$

$$\frac{\partial}{\partial y}\left[x^4 \sin(xy^3)\right] - \frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}(0)$$

$$x^4 \cos(xy^3)\, 3xy^2 - z - y\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-z + 3x^5 y^2 \cos(xy^3)}{y}$$

(b) $yz^3 - \ln yz = x^2 y + y^2 z$

Solution

$$\frac{\partial}{\partial x}\left[yz^3 - \ln yz\right] = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial x}(y^2 z)$$

$$3yz^2\frac{\partial z}{\partial x}-\frac{1}{yz}\left(y\frac{\partial z}{\partial x}\right)=2xy+y^2\frac{\partial z}{\partial x}$$

$$\left(3yz^2-\frac{1}{z}-y^2\right)\frac{\partial z}{\partial x}=2xy$$

$$\frac{\partial z}{\partial x}=\frac{2xy}{3yz^2-\frac{1}{z}-y^2}$$

$$\frac{\partial}{\partial y}\left[yz^3-\ln yz\right]=\frac{\partial}{\partial y}\left(x^2y\right)+\frac{\partial}{\partial y}\left(y^2z\right)$$

$$z^3+3yz^2\frac{\partial z}{\partial y}-\frac{1}{yz}\left(z+y\frac{\partial z}{\partial y}\right)=x^2+2yz+y^2\frac{\partial z}{\partial y}$$

$$\left(3yz^2-\frac{1}{z}-y^2\right)\frac{\partial z}{\partial y}=-z^3+\frac{1}{y}+x^2+2yz$$

$$\frac{\partial z}{\partial y}=\frac{-z^3+\frac{1}{y}+x^2+2yz}{3yz^2-\frac{1}{z}-y^2}$$

(c) $e^{xy}-2e^{xz}+3e^{yz}=x^2yz^3$

Solution:

$$\frac{\partial}{\partial x}\left[e^{xy}-2e^{xz}+3e^{yz}\right]=\frac{\partial}{\partial x}\left(x^2yz^3\right)$$

$$ye^{xy}-2e^{xz}\left(z+x\frac{\partial z}{\partial x}\right)+3ye^{yz}\frac{\partial z}{\partial x}=2xyz^3+3x^2yz^2\frac{\partial z}{\partial x}$$

$$\left(-2xe^{xz}+3ye^{yz}-3x^2yz^2\right)\frac{\partial z}{\partial x}=2xyz^3-ye^{xy}+2ze^{xz}$$

$$\frac{\partial z}{\partial x}=\frac{2xyz^3-ye^{xy}+2ze^{xz}}{-2xe^{xz}+3ye^{yz}-3x^2yz^2}$$

$$\frac{\partial}{\partial y}\left[e^{xy}-2e^{xz}+3e^{yz}\right]=\frac{\partial}{\partial y}\left(x^2yz^3\right)$$

$$xe^{xy}-2e^{xz}x\frac{\partial z}{\partial y}+3e^{yz}\left(z+y\frac{\partial z}{\partial y}\right)=x^2z^3+3x^2yz^2\frac{\partial z}{\partial y}$$

$$\left(-2xe^{xz}+3ye^{yz}-3x^2yz^2\right)\frac{\partial z}{\partial y}=x^2z^3-xe^{xy}-3ze^{yz}$$

$$\frac{\partial z}{\partial y}=\frac{x^2z^3-xe^{xy}-3ze^{yz}}{-2xe^{xz}+3ye^{yz}-3x^2yz^2}$$

Question 11

Use partial derivatives to find an approximation if the values of (*x*, *y*) or (*x*, *y*, z) are changed as stated below.

(a) $f(x, y) = \sqrt{x^2 + 2y^2}$, $(3,6) \to (3.02, 5.99)$

Solution:

$$f_x = \frac{1}{2}\left(x^2 + 2y^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 2y^2}},$$

$$f_x(3,6) = \frac{3}{\sqrt{9 + 2(36)}} = \frac{3}{\sqrt{81}} = \frac{3}{9} = \frac{1}{3}$$

$$f_y = \frac{1}{2}\left(x^2 + 2y^2\right)^{-1/2}(4y) = \frac{2y}{\sqrt{x^2 + 2y^2}}$$

$$f_y(3,6) = \frac{2(6)}{\sqrt{9 + 2(36)}} = \frac{12}{\sqrt{81}} = \frac{12}{9} = \frac{4}{3}$$

$\delta x = 0.02, \quad \delta y = -0.01$

Therefore, $\delta f = f_x \delta x + f_y \delta y = \frac{1}{3}(0.02) + \frac{4}{3}(-0.01) = -0.0067$

(b) $f(x, y, z) = x^2 e^{xyz}$, $(1,1,0) \to (0.97, 1.04, 0.02)$

Solution:

$$f_x = 2xe^{xyz} + x^2yze^{xyz} = e^{xyz}\left(2x + x^2yz\right)$$

$$f_x(1,1,0) = e^0(2 + 0) = 2$$

$$f_y = x^3ze^{xyz} \qquad f_z = x^3ye^{xyz}$$

$$f_y(1,1,0) = 1(0)e^0 = 0 \qquad f_z(1,1,0) = 1e^0 = 1$$

$\delta x = -0.03, \quad \delta y = 0.04, \quad \delta z = 0.02$

Therefore, $\delta f = f_x \delta x + f_y \delta y + f_z \delta z = 2(-0.03) + 0(0.04) + 1(0.02) = -0.04$

Question 12
Use partial derivative to find the approximation values.

(a) $\dfrac{1}{\sqrt{(3.92)^2+(3.01)^2}}$

Solution:

Let $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$

Consider the changes from $(4,3)\rightarrow(3.92,3.01)$

$$f(4,3)=\frac{1}{\sqrt{4^2+3^2}}=\frac{1}{5}$$

$$f_x=-\frac{1}{2}\left(x^2+y^2\right)^{-3/2}(2x)=\frac{-x}{\left(x^2+y^2\right)^{3/2}},$$

$$f_x(4,3)=\frac{-4}{(16+9)^{3/2}}=\frac{-4}{125}$$

$$f_y=-\frac{1}{2}\left(x^2+y^2\right)^{-3/2}(2y)=\frac{-y}{\left(x^2+y^2\right)^{3/2}}$$

$$f_y(4,3)=\frac{-3}{(16+9)^{3/2}}=\frac{-3}{125}$$

$\delta x=-0.08,\quad \delta y=0.01$

Therefore, $\delta f=f_x\delta x+f_y\delta y=\dfrac{-4}{125}(-0.08)+\dfrac{-3}{125}(0.01)=0.00232$

$$\therefore\ \frac{1}{\sqrt{(3.92)^2+(3.01)^2}}\approx f(4,3)+\delta f=\frac{1}{5}+0.00232=0.20232$$

(b) $3.97\sqrt{4(2.04)^3+3.96}$

Solution:

Let $f(x,y,z)=x\sqrt{4y^3+z}$

Consider the changes from $(4,2,4)\rightarrow(3.97,2.04,3.96)$

$f(4,2,4) = 4\sqrt{4(8)+4} = 4\sqrt{36} = 24$

$f_x = \sqrt{4y^3 + z} \qquad f_x(4,2,4) = \sqrt{4(8)+4} = \sqrt{36} = 6$

$f_y = \frac{1}{2}x(4y^3+z)^{-1/2}12y^2 = 6xy^2(4y^3+z)^{-1/2}$

$f_y(4,2,4) = 6(4)(4)(4(8)+4)^{-1/2} = 6 \cdot 16 \cdot \frac{1}{6} = 16$

$f_z = \frac{1}{2}x(4y^3+z)^{-1/2} = \frac{x}{2}(4y^3+z)^{-1/2} \qquad f_z(4,2,4) = \frac{4}{2} \cdot \frac{1}{6} = \frac{1}{3}$

$\delta x = -0.03, \quad \delta y = 0.04, \quad \delta z = -0.04$

Therefore, $\delta f = f_x \delta x + f_y \delta y + f_z \delta z = 6(-0.03) + 16(0.04) + \frac{1}{3}(-0.04) = 0.4467$

$\therefore 3.97\sqrt{4(2.04)^3 + 3.96} \approx f(4,2,4) + \delta f = 24 + 0.4467 = 24.4467$

Question 13

Let $f(x,y,z) = x^3y^2z + 3\cos yz + x^{-2} - \sqrt{2y}$. Estimate the changes of the function if (x,y,z) changes from $(1,2,\frac{\pi}{4})$ to $(0.99, 1.96, \frac{3\pi}{16})$.

<u>Solution</u>

$f(x,y,z) = x^3y^2z + 3\cos yz + x^{-2} - \sqrt{2y}$

The changes in (x,y,z) are $(1,2,\pi/4) \to (0.99, 1.96, 3\pi/16)$,

Thus,

$f_x(x,y,z) = 3x^2y^2z - 2x^{-3}$

$f_x(1,2,\pi/4) = 3(1)(4)(\pi/4) - 2(1) = 3\pi - 2$

$f_y(x,y,z) = 2x^3yz - 3z\sin yz - (2y)^{-1/2}$

$f_y(1,2,\pi/4) = 2(1)(2)(\pi/4) - 3(\pi/4)\sin\pi/2 - 1/2 = \pi/4 - 1/2$

$f_z = x^3y^2 - 3y\sin yz$

$f_z(1,2,\pi/4) = (1)(4) - 3(2)\sin\pi/2 = 4 - 6 = -2$

$\delta x = -0.01, \quad \delta y = -0.04, \quad \delta z = -\pi/16$

Therefore,

$$\delta f = f_x \delta x + f_y \delta y + f_z \delta z = (3\pi - 2)(-0.01) + \left(\frac{\pi}{4} - \frac{1}{2}\right)(-0.04) - 2\left(-\frac{\pi}{16}\right)$$

$$= -0.03\pi + 0.02 - 0.01\pi + 0.02 + 0.125\pi = 0.04 + 0.085\pi$$

Question 14

The length, width and height of a recrangular box are measured to be 4 meter, 3 meter and 6 meter respectively with a maximum possible error of $\frac{50}{168}$ cm. Estimate the maximum possible error in calculating

(a) The area of the surface

Solution

$$A(x, y, z) = 2xy + 2xz + 2yz$$
$$A_x = 2y + 2z, \qquad A_x(4,3,6) = 2(3) + 2(6) = 18$$
$$A_y = 2x + 2z, \qquad A_y(4,3,6) = 2(4) + 2(6) = 20$$
$$A_z = 2x + 2y, \qquad A_z(4,3,6) = 2(4) + 2(3) = 14$$

$$\delta x = \delta y = \delta z = 50/168cm = 1/336m$$

$$\delta A = A_x \delta x + A_y \delta y + A_z \delta z = 18(1/336) + 20(1/336) + 14(1/336) = 52/336 = 13/84m$$

(b) The volume of the box

Solution:

$$V(x, y, z) = xyz$$

$$V_x = yz, \qquad V_x(4,3,6) = 3(6) = 18$$
$$V_y = xz, \qquad V_y(4,3,6) = 4(6) = 24$$
$$V_z = xy, \qquad V_z(4,3,6) = 4(3) = 12$$

$$\delta x = \delta y = \delta z = 50/168cm = 1/336m$$

$$\delta V = V_x \delta x + V_y \delta y + V_z \delta z = 18(1/336) + 24(1/336) + 12(1/336) = 54/336 = 9/56m$$

Question 15

Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(a) $z = u^2 \cos v;\ u = x^3 y^2,\ v = x^2 + y^2$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= 2u\cos v\cdot 3x^2y^2 + u^2(-\sin v)\cdot 2x$$
$$= 2(x^3y^2)\cos(x^2+y^2)3x^2y^2 - (x^3y^2)^2\sin(x^2+y^2)2x$$
$$= 6x^5y^4\cos(x^2+y^2) - 2x^7y^4\sin(x^2+y^2)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$= 2u\cos v\cdot 2x^3y + u^2(-\sin v)\cdot 2y$$
$$= 2(x^3y^2)\cos(x^2+y^2)2x^3y - (x^3y^2)^2\sin(x^2+y^2)2y$$
$$= 4x^6y^3\cos(x^2+y^2) - 2x^6y^5\sin(x^2+y^2)$$

(b) $z = uv^2 - u^3v;\ u = y\sin x,\ v = x\cos y$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$= (v^2 - 3u^2v)\cdot y\cos x + (2uv - u^3)\cdot\cos y$$
$$= \left[(x\cos y)^2 - 3(y\sin x)^2(x\cos y)\right]y\cos x + \left[2(y\sin x)(x\cos y) - (y\sin x)^3\right]\cos y$$
$$= x^2y\cos^2 y\cos x - 3y^3x\sin^2 x\cos y\cos x + 2yx\sin x\cos^2 y - y^3\sin^3 x\cos y$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$
$$= (v^2 - 3u^2v)\cdot\sin x + (2uv - u^3)\cdot(-x\sin y)$$
$$= \left[(x\cos y)^2 - 3(y\sin x)^2(x\cos y)\right]\sin x - \left[2(y\sin x)(x\cos y) - (y\sin x)^3\right]x\sin y$$
$$= x^2\cos^2 y\sin x - 3y^2x\sin^3 x\cos y - 2yx^2\sin x\cos y\sin y + y^3x\sin^3 x\sin y$$

Question 16

Use the chain rule to find $\frac{\partial q}{\partial s}$ if $q = 2x + 2y^2 - 5z^3;$ where

$x = -r + 2s + t - 3u, y = r + t, z = -4s - t + 3u$

Solution:

$$\frac{\partial q}{\partial s} = \frac{\partial q}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial q}{\partial z} \cdot \frac{\partial z}{\partial s} = 2 \cdot 2 + \left(-15z^2\right) \cdot (-4) = 4 + 60(-4s - t + 3u)^2$$

Question 17

Use the chain rule to find $\frac{\partial p}{\partial t}$ if $p = q - \cos r + \sin s;$ where

$q = \sin^2 t, r = t, s = 5t$

Solution:

$$\frac{dp}{dt} = \frac{\partial p}{\partial q} \cdot \frac{dq}{dt} + \frac{\partial p}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial p}{\partial s} \cdot \frac{ds}{dt} = 1 \cdot 2\sin t \cos t + \sin r \cdot 1 + \cos s \cdot 5 = \sin 2t + \sin t + 5\cos 5t$$

Question 18

Assuming that the following equations define y an implicit function of x and $y = f(x)$, find $\dfrac{dy}{dx}$

(a) $x^3 - 2x^2y + 3xy^2 + y^3 = 4$

Solution:

$F(x,y) = x^3 - 2x^2y + 3xy^2 + y^3 - 4 = 0$

Since $F(x,y) = 0$ and $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-(3x^2 - 4xy + 3y^2)}{-2x^2 + 6xy + 3y^2}.$$

(b) $\sqrt{xy} + \sin xy + e^{xy} = 2$

Solution:

$F(x,y) = \sqrt{xy} + \sin xy + e^{xy} = 0$

Since $F(x,y) = 0$ and $y = f(x)$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{-\left[1/2\,y(xy)^{-1/2} + y\cos xy + ye^{xy}\right]}{1/2\,x(xy)^{-1/2} + x\cos xy + xe^{xy}} = -\frac{y}{x}\frac{(xy)^{-1/2} + \cos xy + e^{xy}}{(xy)^{-1/2} + \cos xy + e^{xy}} = -\frac{y}{x}.$$

Question 19

Assuming that the following equations define z an implicit function of x and y while $z = f(x,y)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

(a) $x^3z - 2yz^2 + 3xy^2z + x^2y^3 + z = 4$

Solution:

$F(x,y,z) = x^3z - 2yz^2 + 3xy^2z + x^2y^3 + z - 4 = 0$

Since $F(x,y) = 0$ and $y = f(x)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-\left(3x^2 z + 3y^2 z + 2xy^3\right)}{x^3 - 4yz + 3xy^2 + 1}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-\left(-2z^2 + 6xyz + 3x^2 y^2\right)}{x^3 - 4yz + 3xy^2 + 1}$$

(b) $x^2 y + x + y^3 + e^z - xyz = z^2$

Solution:

$$F(x, y, z) = x^2 y + x + y^3 + e^z - xyz - z^2 = 0$$

Since $F(x, y) = 0$ and $y = f(x)$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-\left(2xy + 1 - yz\right)}{e^z - xy - 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-\left(x^2 + 3y^2 - xz\right)}{e^z - xy - 2z}$$

Question 20

Find the local extremum of the following functions.

(a) $f(x, y) = 2x^3 y - 2x + 16y - 5$

Solution:

1) Find the critical points.

$f_x = 6x^2 y - 2 = 0$

$6x^2 y = 2$

$y = \dfrac{1}{3x^2}$

$y = \dfrac{1}{3(-2)^2} = \dfrac{1}{12}$

$f_y = 2x^3 + 16 = 0$

$2x^3 = -16$

$x^3 = -8, \quad x = -2$

Therefore, the critical point is at $\left(-2, \dfrac{1}{12}\right)$.

2) Determine whether the critical point are minimum, maximum or saddle points.

$f_{xx} = 12xy,$ $\quad f_{xy} = 6x^2,$ $\quad f_{yy} = 0$

$f_{xx}\left(-2,\frac{1}{12}\right) = 12(-2)\frac{1}{12}$ $\quad f_{xy}\left(-2,\frac{1}{12}\right) = 6(-2)^2$ $\quad f_{yy}\left(-2,\frac{1}{12}\right) = 0$

$= -2$ $\quad = 24$

and

$$G\left(-2,\frac{1}{12}\right) = (-2)(0) - (24)^2 = -576.$$

Since, $G\left(-2,\frac{1}{12}\right) < 0$,

Therefore, there is a saddle point at $f\left(-2,\frac{1}{12}\right)$.

(b) $f(x,y) = e^{x^2-y^2+4y}$

<u>Solution:</u>

1) Find the critical points.

$f_x = 2xe^{x^2-y^2+4y} = 0$ $\quad f_y = (-2y+4)e^{x^2-y^2+4y} = 0$

$2x = 0,\ x = 0$ $\quad -2y+4 = 0,\ y = 2$

Thus, the Critical point is at $(0,2)$.

2) Determine whether the critical point are minimum, maximum or saddle points.

$f_{xx} = 2e^{x^2-y^2+4y} + 4x^2e^{x^2-y^2+4y},$ $\quad f_{xx}(0,2) = 2e^4$

$f_{xy} = 2x(-2y+4)e^{x^2-y^2+4y},$ $\quad f_{xy}(0,2) = 0$

$f_{yy} = -2e^{x^2-y^2+4y} + (-2y+4)^2e^{x^2-y^2+4y},$ $\quad f_{yy}(0,2) = -2e^4$

and

$$G(0,2) = 2e^4(-2e^4) - 0 = -4e^8 < 0$$

Since $G(0,2) < 0$,

Thus, there is a saddle point at $f(0,2)$.

(c) $f(x,y) = x^2 + xy + 3x + 2y - 6$

Solution

1) Find the critical points.

$$f_x = 2x + y + 3 = 0, \qquad f_y = x + 2 = 0$$
$$y = -3 - 2x \qquad x = -2$$
$$y = -3 + 4 = 1$$

Therefore, Critical point is at $(-2,1)$.

2) Determine whether the critical point are minimum, maximum or saddle points.

$$f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 0$$

and

$$G(a,b) = 2(0) - 1 = -1 < 0.$$

Since $G(-2,1) < 0$,

Thus, there is a saddle point at $f(-2,1)$.

(d) $f(x,y) = 2y^3 - 6xy - x^2$

Solution

1) Find the critical points.

$$f_x = -6y - 2x = 0,$$
$$2x = -6y,$$
$$x = -3y$$

$$f_y = 6y^2 - 6x = 0$$
$$6y^2 - 6(-3y) = 0$$
$$6y^2 + 18y = 0$$
$$6y(y+3) = 0, \quad y = 0, \ y = -3$$

$$y = 0, \ x = 0, \qquad y = -3, \ x = -3(-3) = 9$$

Therefore, the critical points are at $(0,0)$ and $(9,-3)$.

2) Determine whether the critical point are minimum, maximum or saddle points.

$f_{xx} = -2, \qquad f_{xy} = -6, \qquad f_{yy} = 12y$

For $(0,0)$,

$f_{xx}(0,0) = -2, \qquad f_{xy}(0,0) = -6, \qquad f_{yy}(0,0) = 0$

and

$G(0,0) = (-2)(0) - 36 = -36 < 0$

Since $G(0,0) < 0$,

Thus, there is a saddle point at $f(0,0)$.

For $(9,-3)$,

$f_{xx}(9,-3) = -2, \qquad f_{xy}(9,-3) = -6, \qquad f_{yy}(9,-3) = -36$

and

$G(9,-3) = (-2)(-36) - 36 = 36 > 0$.

Since $G(9,-3) > 0$ and $f_{xx}(9,-3) = -2 < 0$,

Thus, $f(9,-3)$ is a local minimum value.