CHAPTER ONE

FUNCTIONS AND GRAPHS

1.0 Introduction to Functions

In everyday life, many quantities depend on one or more changing variables eg:

(a) plant growth depends on sunlight and rainfall

(b) speed depends on distance travelled and time taken

(c) voltage depends on current and resistance

(d) test marks depend on attitude, listening in lectures and doing tutorials (among many other variables!!)

1.1 Functions

A function is a rule that relates how one quantity depends on other quantities eg.

(a) $V = IR$ where

V = voltage (V) I = current (A) R = resistance (Ω)

(b) $s = \frac{d}{t}$ where

s = speed (m / s) d = distance (m) t = time taken (s)

1.2 Definition of a Function

Whenever a relationship exists between two variables (or quantities) such that for every value of the first, there is only *one* corresponding value of the second, then we say:

the second variable is a ***function*** *of the first variable.*

The first variable is the *independent* variable (usually x), and the second variable is the *dependent* variable (usually y).

The independent variable and the dependent variable are *real numbers*.

Example 1:

We know the equation for the area of a circle from primary school:

$$A = \pi r^2$$

This is a *function* as each value of the independent variable r gives us *one* value of the dependent variable A.

1.3 General Cases

We use x (independent) and y (dependent) variables for general cases.

Example 2:

In the equation $y = 3x + 1$,

y is a function of x, since for each value of x, there is only one value of y.

If we substitute $x = 5$, we get $y = 16$ and no other value.

The values of y we get depend on the values chosen for x.

Therefore, x is the *independent* variable and y is the *dependent* variable.

Example 3

The force F required to accelerate an object of mass 5 kg by an acceleration of a ms^{-2} is given by: $F = 5a$.

Here, F is a function of the acceleration, a.

The *dependent* variable is F and the *independent* variable is a.

1.4 Function Notation

We normally write functions as: $f(x)$ and read this as "function f of x".

We can use other letters for functions. Common ones are $g(x)$ and $h(x)$. But there are also ones like $P(t)$ which could indicate **power** at time t.

Example 4

We often come across functions like: $y = 2x^2 + 5x + 3$

We can write this using function notation:

$f(x) = 2x^2 + 5x + 3$

Function notation is all about **substitution.**

The value of the function $f(x)$ when $x = a$ is written as $f(a)$.

1.5 Domain and Range

Domain: The complete set of possible values of the independent variable is the *domain* of the function.

In plain English, the definition means:

The domain of a function is all the x values which will make the function "work" by outputting real values.

When finding the domain, remember:

- denominator (bottom) of a fraction cannot be zero
- the values under a square root cannot be negative

Range: The complete set of all possible resulting values of the dependent variable is the range of the function.

In plain English, the definition means:

The range of a function is the possible y values of a function.

When finding the range, remember:

- substitute different x-values into y to see what is happening
- make sure you look for minimum and maximum values of y
- draw a sketch!

Because real numbers are used in the domain and range of a function, values that lead to division by zero or to **imaginary numbers** are not included.

1.6 Rectangular Coordinates

A good way of presenting a function is by graphical representation.

Graphs can give a visual picture of the function.

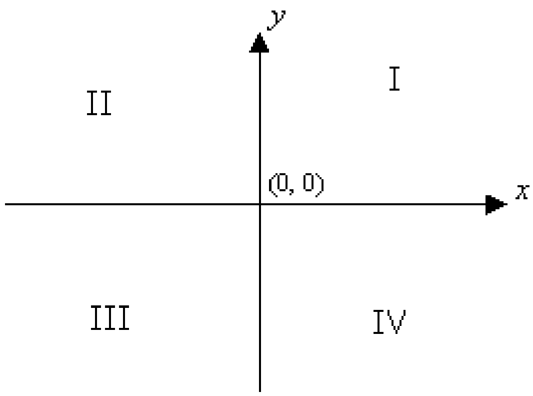
The rectangular co-ordinate system consists of:

the *x*-axis

the *y*-axis

the origin (0,0)

the four quadrants



Normally, the values of the **independent** variable (generally the x-values) are placed on the horizontal axis, while the values of the **dependent** variable (generally the y-values) are placed on the vertical axis.

The x-value, called the **abscissa**, is the perpendicular distance of P from the y-axis.

The y-value, called the **ordinate**, is the perpendicular distance of P from the x-axis.

The values of x and y together, written as (x, y) are called the **co-ordinates** of the point P.

1.7 Graphs of Linear Functions

It is very important to know how to quickly sketch straight lines. This section consists of some reminders for you...

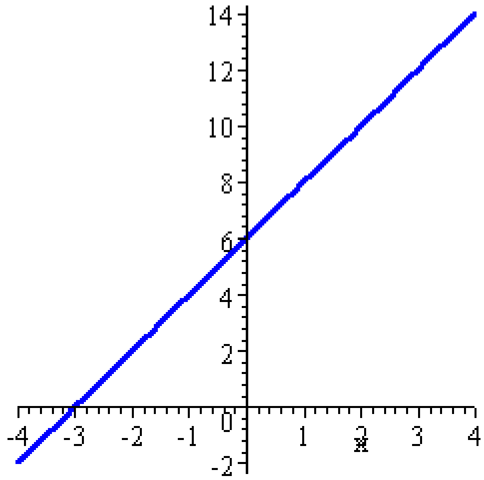
1.7.1 Slope-Intercept Form of a Straight Line: $y = mx + c$

If the slope (also known as gradient) of a line is m, and the y-intercept is c, then the equation of the line is written:

$$y = mx + c$$

Example 5

The line $y = 2x + 6$ has slope $m = 2$ and y-intercept $c = 6$.



1.7.2 Intercept Form of a Straight Line: $ax + by = c$

Often a straight line is written in the form $ax + by = c$. One way we can sketch this is by finding the x- and y-intercepts and then joining those intercepts.

Slope of a Line

The slope (or gradient) of a straight line is given by:

$$m = \frac{\text{vertical rise}}{\text{horizontal run}}$$

We can also write the slope of the straight line passing through the points (x_1, y_1) and (x_2, y_2) as:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Using this expression for slope, we can derive the following.

1.7.3 Point-slope Form of a Straight Line: $y - y_1 = m(x - x_1)$

If a line passes through the point (x_1, y_1) and has slope m, then the equation of the line is given by:

$$y - y_1 = m(x - x_1)$$

1.8 The Straight Line

The slope-intercept form (otherwise known as "gradient, *y*-intercept" form) of a line is given by:

$$y = mx + b$$

This tells us the *slope* of the line is m and the y-intercept of the line is b.

Example 5

The line $y = 2x - 4$ has

- slope $m = 2$ and
- y-intercept $b = -4$.

We notice that this is a *function* - each value of x that we have gives a corresponding value of y.

We need other forms of the straight line as well. A useful form is the **point-slope form** (or point - gradient form)**:**

$$y - y_1 = m(x - x_1)$$

1.9 General Form

Another form of the straight line which we come across is **general form:**

$Ax + By + C = 0$

It can be useful for drawing lines by finding the y-intercept (put $x = 0$) and the x-intercept (put $y = 0$).

Example 6

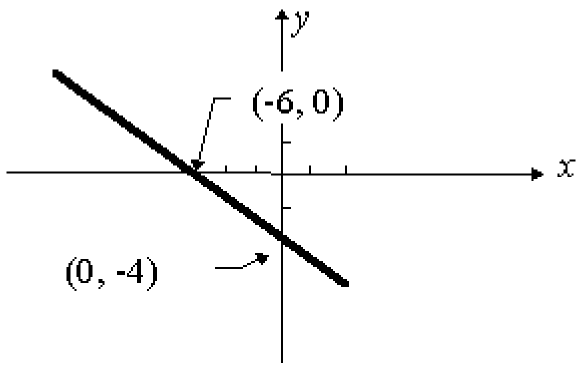
Sketch the line $2x + 3y + 12 = 0$.

Answer

If $x = 0$, we have: $3y + 12 = 0$, so $y = -4$.

If $y = 0$, we have: $2x + 12 = 0$, so $x = -6$.

So the line is:



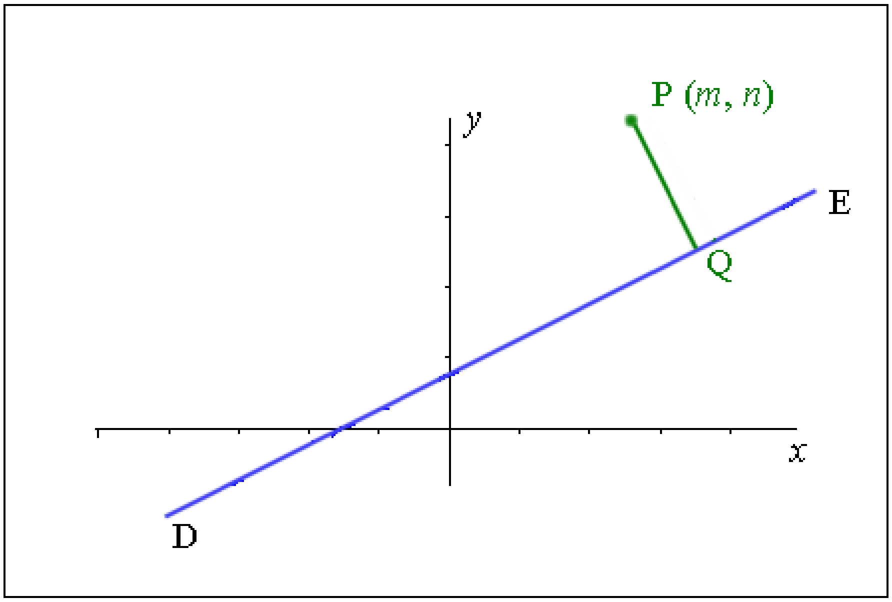
1.10 Perpendicular Distance from a Point to a Line

This is a great problem because it uses all the things we have learned so far:

- distance formula
- slope of parallel and perpendicular lines
- coordinates
- different forms of the straight line

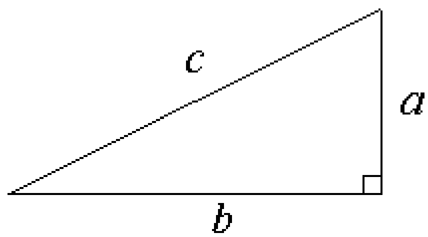
The equation of the line is $Ax + By + C = 0$.

We have a point P with coordinates (m, n). We wish to find the perpendicular distance from the point P to the line (that is, distance PQ).



$$d = \frac{Am + Bn + C}{\sqrt{A^2 + B^2}}$$

1.11 Distance Formula

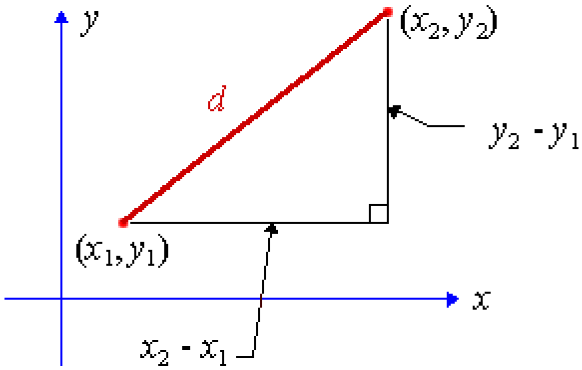


Recall Pythagoras' Theorem:

For a right-angled triangle with *c* at the hypotenuse,

$$c = \sqrt{a^2 + b^2}$$

We use this to find the distance between any two points (x_1, y_1) and (x_2, y_2) on the **cartesian plane**:



The cartesian plane was named after Rene Descartes. It is also called the *x*-*y* plane.

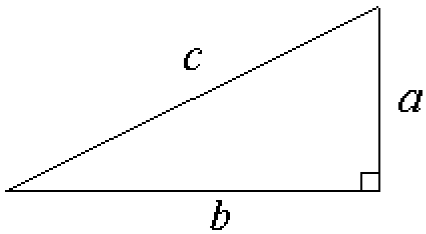
We see that the distance between (x_1, y_1) and (x_2, y_2) is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.12 Gradient (or slope)

The **gradient** of a line is defined as

$$\frac{\text{vertical rise}}{\text{horizontal run}}$$

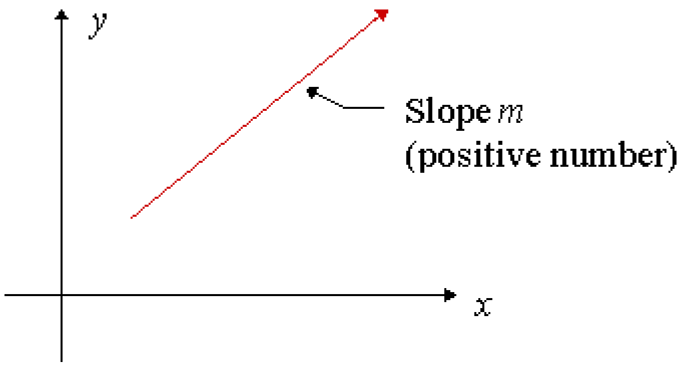


In this diagram, the gradient of the line is given by: $\frac{a}{b}$

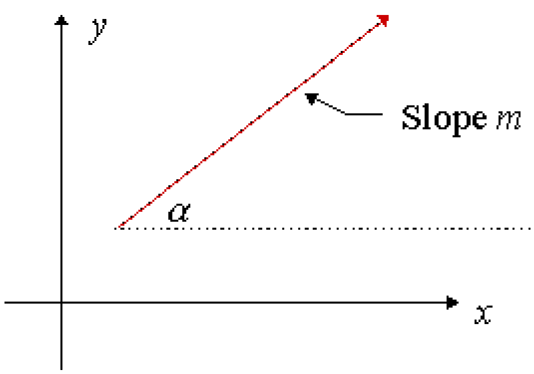
In general, for the line joining the points (x_1, y_1) and (x_2, y_2), we see from the diagram above, that the **gradient** (usually written *m*) is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

NOTE**:** In general, a *positive slope* indicates the value of the dependent variable *increases* as we go left to right:



1.13 Inclination



If the slope of a line is m and the angle the line makes with the x-axis is written α, then (because the tangent of an angle is defined as opposite/adjacent and slope is also defined as opposite/adjacent), we have

$$m = \tan \alpha$$

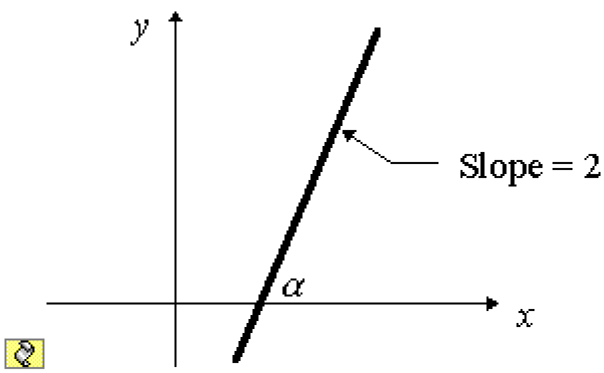
So we can find α using $\alpha = \arctan m$.

This angle α is called the *inclination* of the line.

Example 7

Find the inclination of the line with slope 2.

Answer

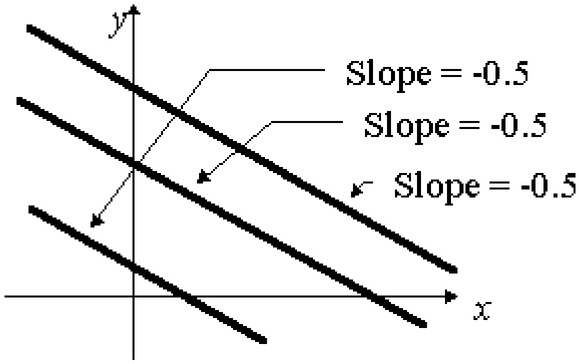


Here, $\tan \alpha = 2$, so

$$\begin{aligned}\alpha &= \arctan(2)\\ &= 63.43^\circ\end{aligned}$$

NOTE**:** The size of angle a is (by definition) only between 0° and 180°.

1.14 Parallel Lines

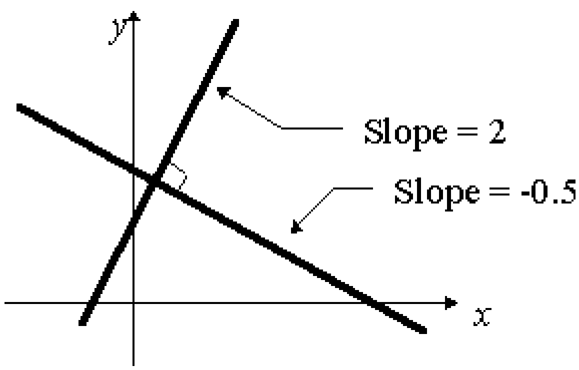


Lines which have the same slope are *parallel.*

If a line has slope m_1 and another line has slope m_2 then the lines are **parallel** if

$$m_1 = m_2$$

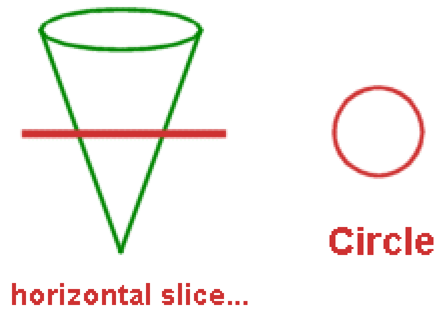
1.15 Perpendicular Lines

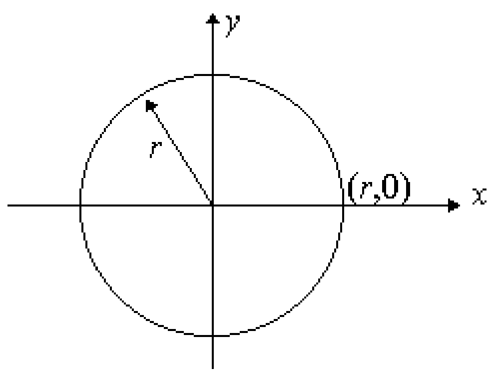


If a line has slope m_1 and another line has slope m_2 then the lines are **perpendicular** if

$$m_1 m_2 = -1$$

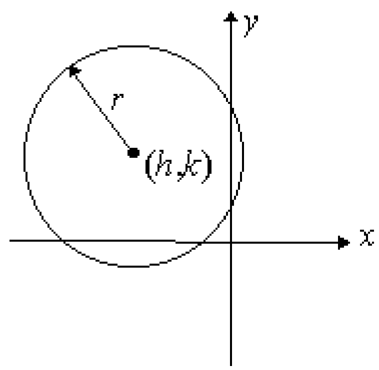
1.16 The Circle





The circle with centre (0, 0) and radius *r* has the equation:

$$x^2 + y^2 = r^2$$



The circle with centre (*h*, *k*) and radius *r* has the equation:

$$(x - h)^2 + (y - k)^2 = r^2$$

1.17 The General Form of the Circle

Any equation which can be written in the form

$$x^2 + y^2 + Dx + Ey + F = 0$$

(with constants *D*, *E, F*)represents a circle.

1.18 The Unit Step Function (Heaviside Function)

In engineering applications, we frequently encounter functions whose values change abruptly at specified values of time *t*. One common example is when a voltage is switched on or off in an electrical circuit at a specified value of time *t*.

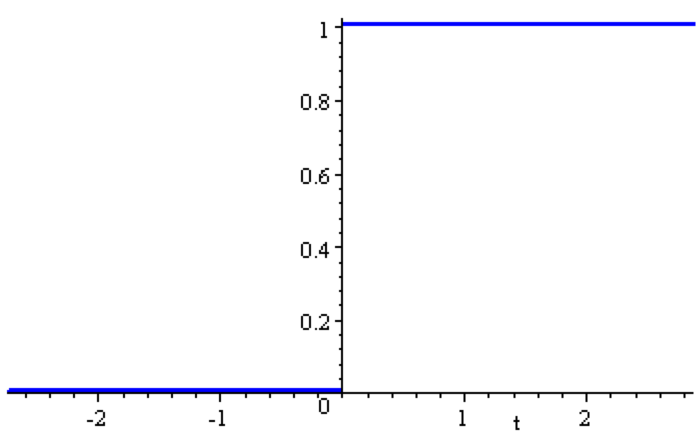
The value of *t* = 0 is usually taken as a convenient time to switch on or off the given voltage. The switching process can be described mathematically by the function called the UNIT STEP FUNCTION (otherwise known as the **Heaviside function** after Oliver Heaviside).

1.19 The Unit Step Function

Definition: The unit step function, ***u*(*t*)**, is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

That is, *u* is a function of time *t*, and *u* has value zero when its argument is negative; and 1 when its argument is positive.



1.20 Shifted Unit Step Function

In many circuits, waveforms are applied at specified intervals other than $t = 0$.. Such a function may be described using the SHIFTED (or "DELAYED") UNIT STEP FUNCTION.

Definition of Shifted Unit Step Function:

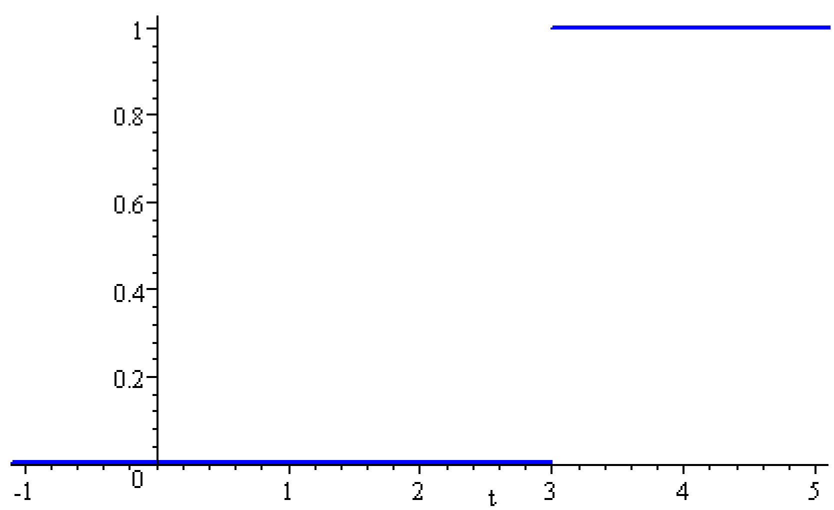
$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

Example 8 (Shifted Unit Step Function)

$f(t) = u(t-3)$

This means $f(t)$ has value of 0 when $t < 3$ and 1 when $t > 3$.

The sketch of the waveform is as follows:



1.21 Rectangular Pulse

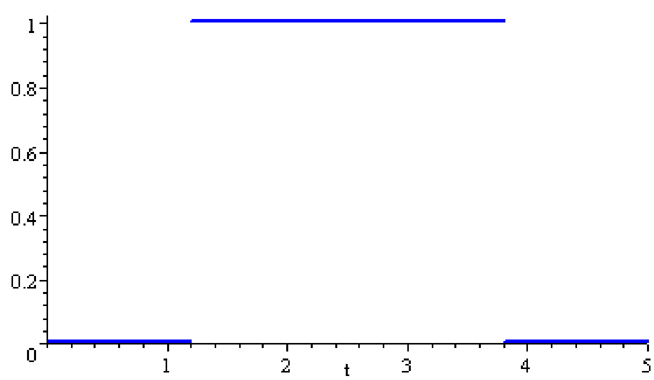
A common situation in a circuit is for a voltage to be applied at a particular time (say $t = a$) and removed later, at $t = b$ (say). This function is written using unit step functions as

$$V(t) = u(t - a) - u(t - b)$$

and it has strength 1, duration $(b - a)$.

Example 9

The graph of $V(t) = u(t - 1.2) - u(t - 3.8)$ is as follows. Here, the duration is $3.8 - 1.2 = 2.6$.



1.22 Products Involving Unit Step Functions

When combined with other functions defined for $t > 0$, the unit step function "turns off" a portion of their graph.

Examples of products with unit function

Note the differences between the following:

$f(t) \bullet u(t)$

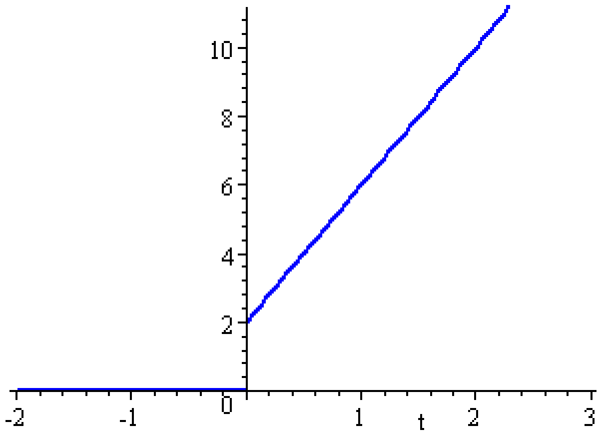
$f(t) \bullet u(t - a)$

Let's see what difference each one makes in some examples.

Example 10

Let $f(t) = 4t + 2$ and $a = 1$.

$g_1(t) = f(t) \bullet u(t) = (4t + 2) \bullet u(t)$



$g_2(t) = f(t) \bullet u(t - a) = (4t + 2) \bullet u(t - 1)$