CHAPTER ONE

FUNDAMENTAL CONCEPTS OF ALGEBRA

Introduction

In this chapter, we will work with numbers. Therefore we need to know about this number and understand its properties. After being introduced to the number systems, we can look at the operation of the numbers such as exponents, surd and logarithm. Algebra is about operations and the order of applying these operations in solving problems in quadratic equations and partial fractions.

Objectives

After completing these tutorials, students should be able to:

- Express each of the given numbers as a quotient $\frac{a}{b}$.
- Write each of the given inequalities in interval notation and show them on the real number line.
- Simplify the given index expressions.
- Solve the following index equations.
- Express in terms of the simplest possible surds.
- Rationalize the denominator and simplify.
- Express in logarithm form.
- Express in index form.
- Simplify the given logarithm expressions.
- Solve the given logarithm equations.

Question 1

Express each of the numbers as a quotient $\frac{a}{b}$

(a) 1.666666666……

Solution:

Let $x = 1.6666\ldots$ (1)

$10x = 16.66\ldots$ (2)

Equation (2)-(1)

$9x = 15$

$$x = \frac{15}{9}$$

$$x = \frac{5}{3}$$

(b) 2.718181818……

Solution:

Let $x = 2.71818\ldots$ (1)

$10x = 27.1818\ldots$ (2)

$1000x = 2718.18..$ (3)

Equation (3)-(2)

$990x = 2691$

$$x = \frac{2691}{990}$$

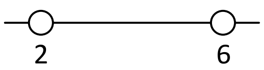
$$x = \frac{299}{110}$$

Question 2

Write each of the following inequalities in interval notation and show them on the real number line.

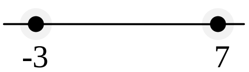
(a) $2 < x < 6$

Solution:



(b) $-3 \le x \le 7$

Solution:

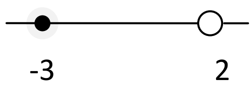


(c) $-2 < x \leq 0$

Solution:

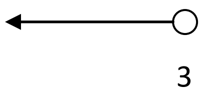
(d) $-3 \leq x < 2$

Solution:



(e) $3 > x$

Solution:



(f) $x \geq -1$

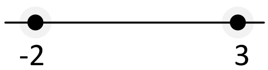
Solution:

Question 3

Show each of the following intervals on the real number line.

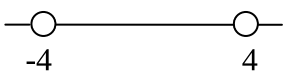
(a) $[-2,3]$

Solution:



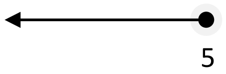
(b) $(-4,4)$

Solution:



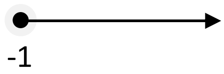
(c) $(-\infty, 5]$

Solution:



(d) $[-1, \infty)$

Solution:



(e) $(-2, 3) \cup (0, 6)$

Solution:

(f) $[-6, 2) \cap (-3, 7)$

Solution:

Question 4

Evaluate

(a) $\left(\frac{1}{2}\right)^{-2}$

Solution:

$$\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^{2}$$
$$= 2^2$$
$$= 4$$

(b) $27^{\frac{-1}{3}}$

Solution:

$$27^{-\frac{1}{3}} = \left(\frac{1}{27}\right)^{\frac{1}{3}}$$

$$= \sqrt[3]{\frac{1}{27}}$$

$$= \frac{1}{3}$$

(c) $\left(\frac{16}{49}\right)^{\frac{1}{2}}$

Solution:

$$\left(\frac{16}{49}\right)^{\frac{1}{2}} = \sqrt{\frac{16}{49}}$$

$$= \frac{4}{7}$$

(d) $(2.56)^{\frac{-1}{2}}$

Solution:

$$(2.56)^{\frac{-1}{2}} = \left(\frac{64}{25}\right)^{\frac{-1}{2}}$$

$$= \left(\frac{25}{64}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{25}{64}}$$

$$= \frac{5}{8}$$

(e) $3^{-1} \cdot 2^2 \cdot 4^0$

Solution:

$$3^{-1} \cdot 2^2 \cdot 4^0 = \frac{1}{3}.4.1$$
$$= \frac{4}{3}$$

(f) $\dfrac{9^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}}{2^{\frac{1}{2}}}$

Solution:

$$\frac{9^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}}{2^{\frac{1}{2}}} = \frac{\left(3^2\right)^{\frac{1}{2}} . \left(2^3\right)^{\frac{1}{2}}}{2^{\frac{1}{2}}}$$
$$= \frac{3.2^{\frac{3}{2}}}{2^{\frac{1}{2}}}$$
$$= 3.2^{\frac{3}{2}-\frac{1}{2}}$$
$$= 3(2)$$
$$= 6$$

Question 5

Simplify the following expressions:

(a) $3^{n+2} \times 9^n \div 27^n$

Solution:

$$3^{n+2} \times 9^n \div 27^n$$
$$= 3^{n+2} \times 3^{2n} \div 3^{3n}$$
$$= 3^{n+2+2n-3n}$$
$$= 3^2$$
$$= 9$$

(b) $4^n \div 8^{\frac{2}{3}n} \times 16^{\frac{1}{4}n}$

Solution:

$4^n \div 8^{\frac{2}{3}n} \times 16^{\frac{1}{4}n}$

$= 2^{2n} \div \left(2^3\right)^{\frac{2}{3}n} \times \left(2^4\right)^{\frac{1}{4}n}$

$= 2^{2n-2n+n}$

$= 2^n$

Question 6

Solve the following equations:

(a) $9^x = 27$

Solution:

$9^x = 27$

$3^{2x} = 3^3$

$2x = 3$

$x = \frac{3}{2}$

(b) $2^{x+1} = \frac{1}{64}$

Solution:

$2^{x+1} = \frac{1}{64}$

$2^{x+1} = \frac{1}{2^6}$

$2^{x+1} = 2^{-6}$

$x + 1 = -6$

$x = -7$

(c) $4^{x+2} = 128^{1-x}$

Solution:

$4^{x+2} = 128^{1-x}$

$\left(2^2\right)^{x+2} = \left(2^7\right)^{1-x}$

$2^{2(x+2)} = 2^{7(1-x)}$

$2x + 4 = 7 - 7x$

$9x = 3$

$x = \frac{1}{3}$

(d) $x^{-\frac{2}{5}} = \frac{1}{16}$

Solution:

$x^{-\frac{2}{5}} = \frac{1}{16}$

$\left(\frac{1}{x}\right)^{\frac{2}{5}} = \frac{1}{2^4}$

$2^4 = x^{\frac{2}{5}}$

$4\,\mathbf{lg}\,2 = \frac{2}{5}\mathbf{lg}\,x$

$\mathbf{lg}\,x = \frac{20\,\mathbf{lg}\,2}{2}$

$\mathbf{lg}\,x = 10\,\mathbf{lg}\,2$

$\mathbf{lg}\,x = \mathbf{lg}\,2^{10}$

$x = 2^{10}$

$x = 1024$

(e) $7^{x^2} - 49^{6-2x} = 0$

Solution:

$7^{x^2} - 49^{6-2x} = 0$

$7^{x^2} = 49^{6-2x}$

$7^{x^2} = \left(7^2\right)^{6-2x}$

$x^2 = 12 - 4x$

$x^2 + 4x - 12 = 0$

$(x-2)(x+6) = 0$

$x = 2, -6$

(f) $(4^x)^x = 4\times 8^x$

Solution:

$$(4^x)^x = 4\times 8^x$$
$$(2^{2x})^x = 2^2\times 2^{3x}$$
$$2^{2x^2} = 2^{2+3x}$$
$$2x^2 - 3x - 2 = 0$$
$$(2x+1)(x-2) = 0$$
$$x = -\frac{1}{2}, 2$$

Question 7

Express in terms of the simplest possible surds:

(a) $\sqrt{50}$

Solution:

$$\sqrt{50} = \sqrt{25\times 2}$$
$$= \sqrt{25}\times\sqrt{2}$$
$$= 5\sqrt{2}$$

(b) $\sqrt{200}$

Solution:

$$\sqrt{200} = \sqrt{100\times 2}$$
$$= \sqrt{100}\times\sqrt{2}$$
$$= 10\sqrt{2}$$

Question 8

Simplify

(a) $\sqrt{2}(3-2\sqrt{2})$

Solution:

$$\sqrt{2}(3-2\sqrt{2})$$
$$= 3\sqrt{2} - 4$$

(b) $(\sqrt{3}-2)(\sqrt{3}-1)$

Solution:

$$(\sqrt{3}-2)(\sqrt{3}-1)$$
$$=3-3\sqrt{3}+2$$
$$=5-3\sqrt{3}$$

(c) $(3\sqrt{3}-2)(3\sqrt{3}+2)$

Solution:

$$(3\sqrt{3}-2)(3\sqrt{3}+2)$$
$$=27+6\sqrt{3}-6\sqrt{3}-4$$
$$=23$$

Question 9

Rationalize the denominator and simplify

(a) $\frac{2}{\sqrt{32}}$

Solution:

$$\frac{2}{\sqrt{32}}$$
$$=\frac{2}{\sqrt{32}}\times\frac{\sqrt{32}}{\sqrt{32}}$$
$$=\frac{2\sqrt{32}}{32}$$
$$=\frac{\sqrt{32}}{16}$$

(b) $\frac{5}{2+\sqrt{5}}$

Solution:

$$\frac{5}{2+\sqrt{5}}$$
$$=\frac{5}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}$$
$$=\frac{10-5\sqrt{5}}{-1}$$
$$=-10+5\sqrt{5}$$

(c) $\frac{2}{2\sqrt{3}-3}$

Solution:

$$\frac{2}{2\sqrt{3}-3}$$
$$=\frac{2}{2\sqrt{3}-3}\times\frac{2\sqrt{3}+3}{2\sqrt{3}+3}$$
$$=\frac{6+4\sqrt{3}}{12-9}$$
$$=\frac{6+4\sqrt{3}}{3}$$
$$=2+\frac{4}{3}\sqrt{3}$$

(d) $\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}$

Solution:

$$\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{2}-1}$$
$$=\frac{\sqrt{2}-1+\sqrt{2}+1}{2-1}$$
$$=\frac{2\sqrt{2}}{1}$$
$$=2\sqrt{2}$$

Question 10

Express in logarithm form

(a) $5^3=125$

Solution:

$$5^3=125$$
$$\mathbf{lg}_5\,125=3$$

(b) $1331 = (121)^{\frac{3}{2}}$

Solution:

$1331 = \mathbf{(}121\mathbf{)}^{\frac{3}{2}}$

$\mathbf{lg}_{121} 1331 = \dfrac{3}{2}$

(c) $e^y = 33$

Solution:

$e^y = 33$

$\mathbf{ln}\, 33 = y$

Question 11

Express in index form

(a) $\log_3 27 = 3$

Solution:

$\mathbf{log}_3 27 = 3$

$27 = 3^3$

(b) $\log_x y = 2$

Solution:

$\mathbf{log}_x y = 2$

$y = x^2$

(c) $\ln b = -3$

Solution:

$\mathbf{ln}\, b = -3$

$b = e^{-3}$

Question 12

Evaluate

(a) $\log_9 9^{\frac{1}{2}}$

Solution:

$\mathbf{log}_9 9^{\frac{1}{2}}$

$= \frac{1}{2}\mathbf{log}_9 9$

$= \frac{1}{2}$

(b) $\log_{\frac{1}{2}} 4$

Solution:

$\mathbf{log}_{\frac{1}{2}} 4$

$= \mathbf{log}_{\frac{1}{2}} \left(\frac{1}{4}\right)^{-1}$

$= \mathbf{log}_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-2}$

$= -2\,\mathbf{log}_{\frac{1}{2}} \frac{1}{2}$

$= -2$

Question 13

Express in terms of log a, log b and log c

(a) $\log abc$

Solution:

$\mathbf{log}\, abc$

$= \mathbf{log}\, a + \mathbf{log}\, b + \mathbf{log}\, c$

(b) $\log \frac{a}{b^2 c}$

Solution:

$\mathbf{log} \frac{a}{b^2 c}$

$= \mathbf{log}\, a - \mathbf{log}\, b^2 - \mathbf{log}\, c$

$= \mathbf{log}\, a - 2\,\mathbf{log}\, b - \mathbf{log}\, c$

(c) $\log \sqrt{\frac{a}{b}}$

Solution:

$\mathbf{log} \sqrt{\frac{a}{b}}$

$= \mathbf{log} \left(\frac{a}{b}\right)^{\frac{1}{2}}$

$= \frac{1}{2} \mathbf{log} \left(\frac{a}{b}\right)$

$= \frac{1}{2} \mathbf{log}\, a - \frac{1}{2} \mathbf{log}\, b$

(d) $\ln\left(\frac{a^3}{b}\right)$

Solution:

$\mathbf{ln}\left(\frac{a^3}{b}\right)$

$= \mathbf{ln}\, a^3 - \mathbf{ln}\, b$

$= 3\,\mathbf{ln}\, a - \mathbf{ln}\, b$

Question 14

Simplify

(a) $3\ln(x+7) - \ln x$

Solution:

$3\mathbf{ln}(x+7) - \mathbf{ln}\, x$

$= \mathbf{ln}(x+7)^3 - \mathbf{ln}\, x$

$= \mathbf{ln}\left(\frac{(x+7)^3}{x}\right)$

(b) $\log 2 + \log 6 - \log 4$

Solution:

$\mathbf{log}\, 2 + \mathbf{log}\, 6 - \mathbf{log}\, 4$

$= \mathbf{log}\left(\frac{2\times 6}{4}\right)$

$= \mathbf{log}\, 3$

(c) $\frac{1}{2}\log 25 - 2\log 3 + 2\log 6$

Solution:

$\frac{1}{2}\mathbf{log}\, 25 - 2\mathbf{log}\, 3 + 2\mathbf{log}\, 6$

$= \mathbf{log}\, 25^{\frac{1}{2}} - \mathbf{log}\, 3^2 + \mathbf{log}\, 6^2$

$= \mathbf{log}\, 5 - \mathbf{log}\, 9 + \mathbf{log}\, 36$

$= \mathbf{log}\left(\frac{5\times 36}{9}\right)$

$= \mathbf{log}\, 20$

(d) $\frac{\log 9}{\log 3}$

Solution:

$\frac{\mathbf{log}\, 9}{\mathbf{log}\, 3}$

$= \frac{\mathbf{log}\, 3^2}{\mathbf{log}\, 3}$

$= \frac{2\,\mathbf{log}\, 3}{\mathbf{log}\, 3}$

$= 2$

Question 15

Solve each equation

(a) $\log_4(x^2-9)-\log_4(x+3)=3$

Solution:

$$\log_4(x^2-9)-\log_4(x+3)=3$$

$$\log_4\left(\frac{x^2-9}{x+3}\right)=3\log_4 4$$

$$\log_4\left(\frac{x^2-9}{x+3}\right)=\log_4 4^3$$

$$\frac{x^2-9}{x+3}=4^3$$

$$x^2-9=64(x+3)$$

$$x^2-64x-201=0$$

$$(x-67)(x+3)=0$$

$$x=67,-3$$

(b) $\log_2(x^2+1)-\log_4 x^2=1$

Solution:

$$\log_2(x^2+1)-\log_4 x^2=1$$

$$\log_2(x^2+1)-\frac{\log_2 x^2}{\log_2 4}=1$$

$$\log_2(x^2+1)-\frac{\log_2 x^2}{2}=1$$

$$2\log_2(x^2+1)-\log_2 x^2=2$$

$$\log_2\left(\frac{(x^2+1)^2}{x^2}\right)=2$$

$$\frac{(x^2+1)^2}{x^2}=4$$

$$(x^2+1)^2=4x^2$$

$$x^4+2x^2+1=4x^2$$

$$x^4-2x^2+1=0$$

$$(x^2-1)(x^2-1)=0$$

$$x=1,-1$$