CHAPTER 1 LIMITS AND CONTINUITY

1.1 LIMITS (page 2)

Limits are used to explain changes that arise for a particular function when the value of an independent variable approaches a certain value.

$f(x) = \dfrac{\sin x}{x}$, x is in radians and $x \neq 0$

What will happen to the value of $f(x)$ as x moves along the x-axis approaching x=0?

(a) Table shows x-values approaching 0 from right hand side

	●	←	←	←	←	←
x	**1**	**0.0001**	**0.001**	**0.01**	**0.1**	**1**
***f*(*x*)**	**?**	**0.9999999**	**0.9999998**	**0.9999833**	**0.9983342**	**0.8414709**

When x **approaches 0** from **right hand side**, the value of $f(x) = \dfrac{\sin x}{x}$ **approaches 1**, this limits is written as

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \qquad \textbf{(Right limit)}$$

(b) Table shows *x*-values approaching 0 from left side

	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	•
x	**-1.0**	**-0.1**	**-0.01**	**-0.001**	**-0.0001**	**1**
$f(x)$	**0.8414709**	**0.9983342**	**0.9999833**	**0.9999998**	**0.9999999**	**?**

***f*(*x*) tends to 1** when *x* **approaches 0** form the **left hand side** and is written as

$$\lim_{x \to 0^-} \frac{\sin x}{x} = 1 \qquad \textbf{(Left Limit)}$$

(c) summary from (a) and (b)

Since the limits of $f(x)$ from the right hand side and the left hand side are the **same** and **equal to 1**, we can write

$$\lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

therefore $$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Graph of $f(x) = \dfrac{\sin x}{x}, \qquad x \neq 0$ (page 5)

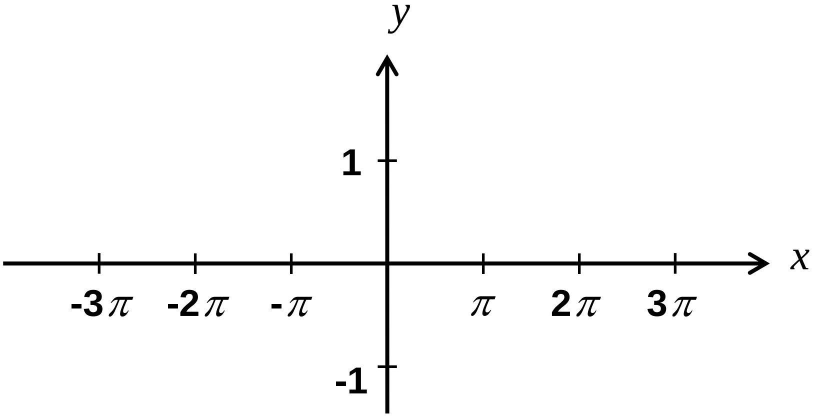


Figure 1.2

Definition 1.1 (Right Limit) – one-sided limit (page 3)

If the value of $f(x)$ tends to a number l_1 as x approaches x_0 from the right hand side, then we write

$$\lim_{x \to x_0^+} f(x) = l_1$$

and we say "limit of $f(x)$ as x approaches x_0 from the right equals l_1"

Definition 1.2 (Left Limit) – one-sided limit (page 4)

If the value of $f(x)$ tends to a number l_2 as x approaches x_0 from the left hand side, then we write

$$\lim_{x \to x_0^-} f(x) = l_2$$

Definition 1.3 (Limit of a Function) (page 5)

If the limit from the left and the right hand side of $f(x)$ has the **same value** i.e

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = l$$

then $\lim_{x \to x_0} f(x)$ **exist** and it is written as $\lim_{x \to x_0} f(x) = l$

However, if either the limit from the left or that from the right does not exist,

or if $$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$$

then $\lim_{x \to x_0} f(x)$ **does not exist**.

Example & Exercise: (another transparency)

Exercise at home: ***(tutorial 1)***

Do Quiz 1 A, no. 1 – 4 (page 10)

Notes:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

(see Example 1.5, page 8)

1.2 CASES OF NON-EXISTENT LIMITS (page 12)

Example 1.9 (page 14):

Sketch the graph $y = \frac{2}{x}$. Hence evaluate $\lim_{x \to 0} \frac{2}{x}$.

Solution:

This function is not determine at x = 0. For $x \to 0$,

	→	→	→	→	•	←	←	←	←
x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$f(x)$	-2	-20	-200	-2000	?	2000	200	20	2

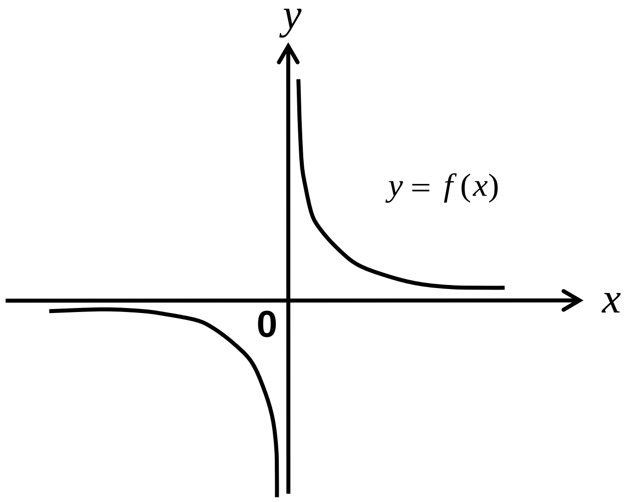


Figure 1.10

As x approaches 0 from the left hand side, the value of $f(x)$ becomes smaller and decreases indefinitely. Hence $f(x)$ does not approaches a fixed definite value.

$\lim_{x \to 0^-} \frac{2}{x}$ does not exist and $\lim_{x \to 0^-} \frac{2}{x} = -\infty$

As x approaches 0 from the right hand side, the value of $f(x)$ becomes bigger and increases indefinitely. Hence $f(x)$ also does not approaches a fixed definite value.

$\lim_{x \to 0^+} \frac{2}{x}$ does not exist and $\lim_{x \to 0^+} \frac{2}{x} = +\infty$

Both values of the limits have different sign,

then $\lim_{x \to 0} \frac{2}{x}$ does not exist.

Exercise at home: ***(tutorial 1)***

Do Quiz 1 B, no. 1 – 3 (page 15)

1.3 LIMITS AT INFINITY (page 17)

Example 1.12 (page 18):

Sketch the graph $y = \frac{2}{x}$. Hence evaluate $\mathrm{had}_{x \to +\infty} \frac{2}{x}$ and $\lim_{x \to -\infty} \frac{2}{x}$.

Solution:

For $x \to \pm\infty$

	←	←	←	←	•	→	→	→	→
x	**-1000**	**-100**	**-10**	**-1**	**0**	**1**	**10**	**100**	**1000**
$f(x)$	**-0.002**	**-0.02**	**-0.2**	**-2**	**?**	**2**	**0.2**	**0.02**	**0.002**

Graph same as Figure 1.10.

As x approaches $+\infty$, the value of $f(x)$ tends to zero.

$$\lim_{x \to +\infty} \frac{2}{x} = 0$$

As x approaches $-\infty$, the value of $f(x)$ also tends to zero.

$$\lim_{x \to -\infty} \frac{2}{x} = 0$$

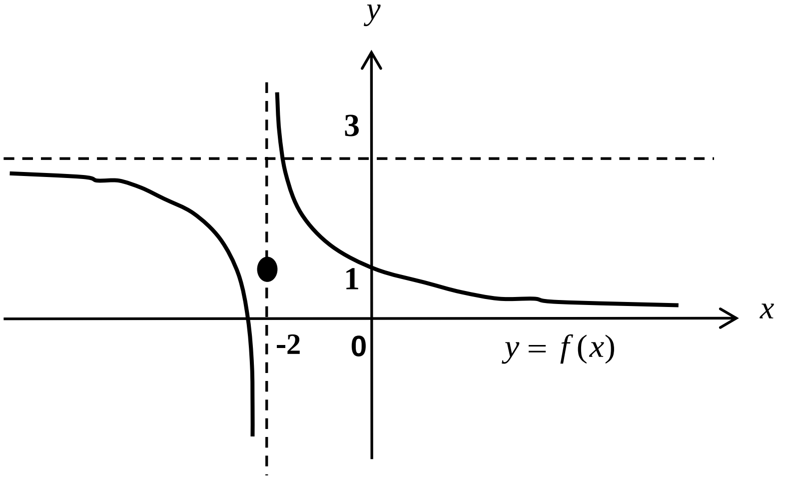
Note:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e \qquad \lim_{x \to 0} (1 + x)^{1/x} = e$$

(See Example 1.14, page 18)

Exercise:

By referring to the figure below, find



a) $\lim_{x \to -2^-} f(x)$ b) $\lim_{x \to -2^+} f(x)$ c) $\lim_{x \to -2} f(x)$

d) $f(-2)$ e) $\lim_{x \to -\infty} f(x)$ f) $\lim_{x \to +\infty} f(x)$

Exercise at home: (tutorial 1)

Do Quiz 1 C, no. 1 – 4 (page 20)

Exercise at home: (tutorial 1)

Do EXERCISE 1, no. 1 – 4, 7 (page 46)

1.4 COMPUTATIONAL METHODS OF LIMIT (page 21)

Theorem 1.1 (Basic Properties of Limits) (page 22)

Suppose a, k and n are real numbers. The limit of each of these expressions is as follows

(a) $\lim_{x \to a} k = k$

(b) $\lim_{x \to a} x = a$

(c) $\lim_{x \to a} x^n = a^n$

Example (page 23):

(a) $\lim_{x \to 5} 3$; $\lim_{x \to +\infty} 3$

(b) $\lim_{x \to -2} x$; $\lim_{\delta x \to 0} \delta x$; $\lim_{t \to \infty} t$

(c) $\lim_{\delta x \to -\frac{1}{2}} (\delta x)^3$; $\lim_{t \to +\infty} t^4$; $\lim_{y \to -\infty} y^5$

(Corrections on the book)

Notes: (extra)

$$\lim_{x \to +\infty} x^n = +\infty, \qquad n = 1, 2, 3, \ldots$$

$$\lim_{x \to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots(odd) \\ +\infty, & n = 2, 4, 6, \ldots(even) \end{cases}$$

Example:

a) $\lim_{x \to +\infty} x^3$ b) $\lim_{x \to +\infty} x^6$ c) $\lim_{x \to -\infty} x^5$ d) $\lim_{x \to -\infty} x^4$

Theorem 1.2 (Properties of Limits) (page 24)

Let $f(x)$ and $g(x)$ be two functions. If the limits of $f(x)$ and $g(x)$ exist at a particular point, then

(a) $\lim [kf(x)] = k \lim f(x)$; k is a constant

(b) $\lim [f(x) \pm g(x)] = \lim f(x) \pm \lim g(x)$

(c) $\lim [f(x) \cdot g(x)] = \lim f(x) \cdot \lim g(x)$

(d) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$, with condition $\lim g(x) \neq 0$

(e) $\lim [f(x)]^n = [\lim f(x)]^n$

(f) $\lim \sqrt[n]{f(x)} = \sqrt[n]{\lim f(x)}$, with condition $\lim f(x) \geq 0$ if n is even

Example (page 24 & 25):

(a) $\lim_{x \to -2} 5x - 5$; $\lim_{y \to -2} \sqrt{5y^2 - 4}$

(b) $\lim_{t \to 1} \dfrac{t-1}{t+3}$; $\lim_{x \to 2} \dfrac{x^3 + 8}{x+2}$

Exercise:

Evaluate the limits of the following expressions

a) $\lim\limits_{x \to 1} (\sqrt{3x-1} + 4)$ b) $\lim\limits_{x \to 4} \left(\dfrac{3x^2 + x - 2}{2x^3 + 5x + 2} \right)$

Remarks:

If both the numerator and denominator of a rational functions equal to zero as x approaches a, this means:

(a) The numerator and denominator have a common factor $(x-a)$ that must be cancelled off before the limits are evaluated. (page 25)

(b) If involve the $\sqrt{\ }$ term, multiply the numerator and denominator with its conjugate.

(c) If involve the trigonometric function, use the result

$\lim\limits_{x \to 0} \dfrac{\sin\ x}{x} = 1$, $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$ and

$\lim\limits_{kx \to 0} \dfrac{\sin\ kx}{kx} = \lim\limits_{x \to 0} \dfrac{\sin\ kx}{kx} = 1$ (page 27)

Example for (a) (page 26):

$$\lim_{x \to -3} \frac{x^2 + 6x + 9}{x + 3}$$

Example for (b) (page 26):

$$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}$$

Example for (c) (page 26 & 28):

$$\lim_{x\to 0}\frac{\sin 3x}{2x} \quad ; \quad \lim_{x\to 0}\frac{x}{\sin x}$$

Exercise:

Evaluate the following limits.

a) $\lim_{x\to 4}\left(\frac{x^2-16}{x-4}\right)$ b) $\lim_{x\to 0}\left(\frac{x}{\sqrt{x+1}-1}\right)$ c) $\lim_{h\to 0}\frac{\sin 2h}{\sin 3h}$

Remarks (page 30):

To evaluate the limits of a rational function $f(x)$ when x approaches positive or negative infinity,

$$\lim_{x\to +\infty} f(x) \quad \text{or} \quad \lim_{x\to -\infty} f(x)$$

Step 1: Divide the numerator and denominator of *f(x)* with x^n,where *n* is the highest power of *x* in the denominator's term.

Step 2: Use the limits theorem.

Example (page 31):

$$\lim_{x\to +\infty}\frac{3x-5}{6x+8} \quad ; \quad \lim_{x\to -\infty}\frac{\sqrt{3x^4+x}}{x^2-8}$$

Exercise:

Evaluate the limits of each of the following expressions.

a) $\lim\limits_{s \to +\infty} \dfrac{5s^2 - 7}{3s^3 + s}$ b) $\lim\limits_{y \to +\infty} \sqrt{\dfrac{4y^2 - 1}{y^2 + 1}}$

Exercise at home: ***(tutorial 2)***

Do Quiz 1 D, no. 1d), 1e), 3a), 3b), 4a), 4c), 5a), 5c) (page 32)

1.6 CONTINUITY (page 40)

Definition 1.5 (Continuity) (page 41)

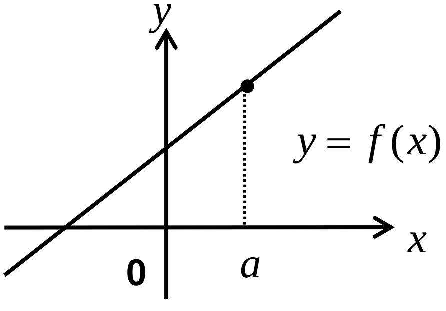
A function $f(x)$ is said to be **continuous at the point** $\boldsymbol{x = a}$ if the following conditions are satisfied.

1. The function $f(x)$ is defined at $x = a$, that is $f(a)$ exist.
2. $\lim\limits_{x \to a} f(x)$ exist.
3. $\lim\limits_{x \to a} f(x) = f(a)$.

Note (page 41):

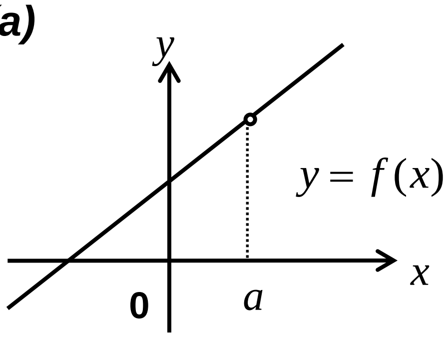
If one or more of the above conditions is not satisfied, then the function $f(x)$ is said to be **not continuous at** $x = a$, and $x = a$ is known as a **point of discontinuity**.

(page 40)



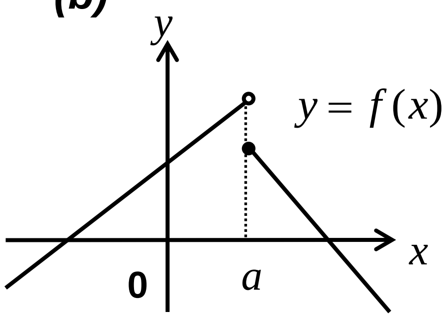
The value of $f(x)$ exist for every value of x. In other words, $f(x)$ is defined for every value of x.

(a)



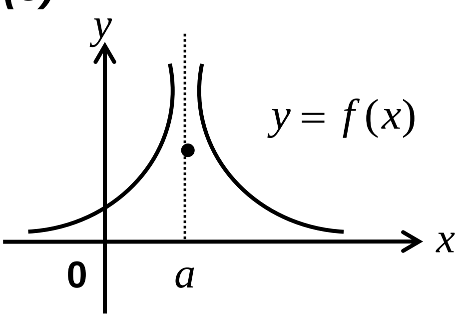
The curve has a hole at the point $x = a$, $f(x)$ is not defined at that point

(b)



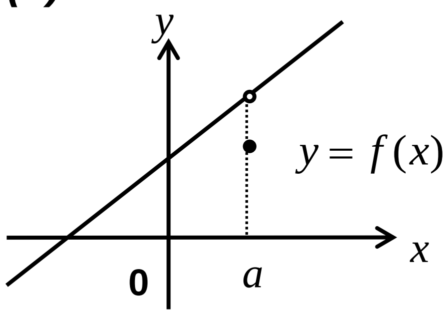
The function $f(x)$ is defined but $\lim_{x \to a} f(x)$ does not exist

(c)



The function $f(x)$ is defined but $\lim_{x \to a} f(x)$ does not exist

(d)



The function $f(x)$ is defined at $x = a$ and $\lim_{x \to a} f(x)$ exist but $\lim_{x \to a} f(x) = f(a)$

Remarks: (page 44)

The followings are a few basic results about continuity:

If $f(x)$ and $g(x)$ are continuous functions at $x = a$, then

i. $f(x) \pm g(x)$ are continuous at $x = a$

ii. $f(x) \cdot g(x)$ is continuous at $x = a$

iii. $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$, if $g(x) \neq 0$, and

discontinuous at $x = a$ if $g(a) = 0$

Example (page 42)

Find the points of discontinuity, if they exist, for the following functions.

$f(x) = x^2$; $f(x) = |x|$; $f(x) = \dfrac{1}{x}$;

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x > 0 \end{cases}$$

Exercise:

Given

$$f(x) = \begin{cases} -1, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases}$$

Find the $\lim\limits_{x \to 0} f(x)$. Is this function continuous at $x = 0$?

Exercise at home: (tutorial 2)

Do Quiz 1 F, no. 1 (page 45)

Exercise at home: (tutorial 2)

Do EXERCISE 1, no. 5, 8b), 8e), 9a), 10a), 11a), 12a), 12c), 13a), 13e), 14a), 14c), 15, 18b), 18c), 19a), 19b), 19f) (page 46-49)
