CHAPTER ONE

SECOND ORDER LINEAR DIFFERENTIAL EQUATION

After completing these tutorials, students should be able to:

- find the general solution of the given homogeneous differential equations
- solve homogeneous differential equations of the given initial value problem
- find the general solution of the given non-homogeneous differential equations by using the method of undetermined coefficients
- solve non-homogeneous differential equations of the given initial value problem by using the method of undetermined coefficients
- find the general solution of the given non-homogeneous differential equations by using the method of variation of parameters
- solve non-homogeneous differential equations of the given initial value problem by using the method of variation of parameters

Question 1

Find the general solution of the given differential equation.

(a) $y'' - 2y' + y = 0$

Solution:

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1$

Real but repeated roots.

$\therefore y = Ae^x + Bxe^x$

(b) $y'' + 5y' + 6y = 0$

Solution:

$m^2 + 5m + 6 = 0$

$(m+3)(m+2) = 0$

$m = -3$ or $m = -2$

Real and different roots

$\therefore y = Ae^{-3x} + Be^{-2x}$

(c) $2y'' + 2y' + y = 0$

Solution:

$2m^2 + 2m + 1 = 0$

$$m = \frac{-2 \pm \sqrt{2^2 - 4(2)(1)}}{2(2)}$$

$$= -\frac{1}{2} \pm \frac{1}{2}i$$

Complex conjugate roots ($a \pm bi$)

$$\therefore y = e^{-\frac{1}{2}x}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$$

(d) $y'' + 9y = 0$

Solution:

$m^2 + 9 = 0$

$m^2 = -9$

$m = \pm\sqrt{-9}$

$= \pm 3i$

Complex conjugate roots ($a \pm bi$)

$\therefore y = A\cos 3x + B\sin 3x$

Question 2

Solve differential equations of the given initial problem.

(a) $9y''-12y'+4y=0$; $y(0)=2$; $y'(0)=-1$

Solution:

$9m^2-12m+4=0$

$(3m-2)(3m-2)=0$

$m=\frac{2}{3}$

Real but repeated roots

$\therefore y=Ae^{\frac{2}{3}x}+Bxe^{\frac{2}{3}x}$

$y(0)=2$

$y(0)=Ae^0+0=2$

$A=2$

$y'(0)=-1$

$y'=\frac{2}{3}Ae^{\frac{2}{3}x}+Be^{\frac{2}{3}x}+\frac{2}{3}Bxe^{\frac{2}{3}x}$

$y'(0)=\frac{2}{3}Ae^0+Be^0+0=-1$

$\frac{2}{3}A+B=-1$

$\frac{2}{3}(2)+B=-1$

$B=-\frac{7}{3}$

$\therefore y(x)=2e^{\frac{2}{3}x}-\frac{7}{3}xe^{\frac{2}{3}x}$

(b) $y''+y'=0$; $y(0)=2$; $y'(0)=1$

Solution:

$m^2+m=0$

$m(m+1)=0$

$m=0$ or $m=1$

Real and different roots

$\therefore\ y = A + Be^{-x}$

$y(0) = 2$

$y(0) = A + Be^{-(0)}$

$2 = A + B$

$A = 2 - B \rightarrow (eq1)$

$y'(0) = 1$

$y'(x) = -Be^{-x}$

$y'(0) = -B$

$1 = -B$

$B = -1 \rightarrow (eq2)$

Substitute ($eq2$) into ($eq1$)

$A = 2 - (-1)$

$A = 3$

$\therefore\ y(x) = 3 - e^{-x}$

(c) $z'' - 2z' - 2z = 0$; $z(0) = 0$; $z'(0) = 3$

Solution:

$m^2 - 2m - 2 = 0$

$$m = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$m = 2.732$ or $m = -0.732$

Real and different roots

$\therefore\ Z = Ae^{2.732x} + Be^{-0.732x}$

$Z(0) = 0$

$Z(0) = A + B$

$0 = A + B$

$A = -B \rightarrow (eq1)$

$Z'(0) = 3$

$Z'(x) = 2.732Ae^{2.732x} - 0.732Be^{-0.732x}$

$Z'(0) = 2.732A - 0.732B$

$3 = 2.732A - 0.732B \rightarrow (eq2)$

Substitute $(eq1)$ into $(eq2)$

$3 = 2.732(-B) - 0.732B$

$B = -0.866$

$A = -(-0.866)$

$A = 0.866$

$Z(x) = 0.866e^{2.732x} - 0.866e^{-0.732x}$

Question 3

Find the general solution of the given non-homogeneous differential equation by using the method of undetermined coefficients

(a) $y'' - 2y' - 3y = 3e^{2x}$

Solution:

$m^2 - 2m - 3 = 0$

$(m - 3)(m + 1) = 0$

$m = 3$ or $m = -1$

Real and different roots

$\therefore y_h = Ae^{3x} + Be^{-x}$

$f(x) = 3e^{2x}$

$y_k = ce^{2x} \rightarrow (eq1)$

$y'_k = 2ce^{2x} \rightarrow (eq2)$

$y''_k = 4ce^{2x} \rightarrow (eq3)$

Substitute $(eq1)$, $(eq2)$ and $(eq3)$ into $y'' - 2y' - 3y = 3e^{2x}$

$4ce^{2x} - 2(2ce^{2x}) - 3(ce^{2x}) = 3e^{2x}$

$-3ce^{2x} = 3e^{2x}$

Equate the coefficinet :

$-3c = 3$

$c = -1$

$\therefore\ y = Ae^{3x} + Be^{-x} - e^{2x}$

(b) $y''+2y'+5y = 3\sin 2x$

<u>Solution:</u>

$m^2 + 2m + 5 = 0$

$$m = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

$m = -1 \pm 2i$

Complex conjugate roots $(a \pm bi)$

$\therefore\ y_h = e^{-h}(A\cos 2x + B\sin 2x)$

$f(x) = 3\sin 2x$

$y_k = p\cos 2x + q\sin 2x \rightarrow (eq1)$

$y'_k = -2p\sin 2x + 2q\sin 2x \rightarrow (eq2)$

$y''_k = -4p\sin 2x - 4q\sin 2x \rightarrow (eq3)$

Substitute $(eq1)$, $(eq2)$ and $(eq3)$ into $y'' + 2y' + 5y = 3\sin 2x$

$-4p\sin 2x - 4q\sin 2x + 2(-2p\sin 2x + 2q\sin 2x) + 5(p\cos 2x + q\sin 2x) = 3\sin 2x$

Equate the coefficient :

$\sin 2x$:

$-4q - 4p + 5q = 3$

$q - 4p = 3 \rightarrow (eq4)$

$\cos 2x$:

$-4p + 4q + 5p = 0$

$p + 4q = 0$

$p = -4q \rightarrow (eq5)$

Substitute (eq5) into $(eq4)$:

$q - 4(-4q) = 3$

$q + 16q = 3$

$17q = 3$

$q = \frac{3}{17}$

$p = -4(\frac{3}{17})$

$p = -\frac{12}{17}$

$\therefore\ y = e^{-x}(A\cos 2x + B\sin 2x) - \frac{12}{17}\cos 2x + \frac{3}{17}\sin 2x$

(c) $2y'' + 3y' + y = x^2 + 3\sin x$

<u>Solution:</u>

$2m^2 + 3m + 1 = 0$

$(2m+1)(m+1) = 0$

$m = -\frac{1}{2}$ or $m = -1$

Roots are real and different

$\therefore\ y_h = Ae^{-\frac{x}{2}} + Be^{-x}$

$f(x) = x^k + 3\sin x$

$y_k = a_2x^2 + a_1x + a_0 + p\cos x + q\sin x \rightarrow (eq1)$

$y'_k = 2a_2x + a_1 - p\sin x + q\cos x \rightarrow (eq2)$

$y''_k = 2a_2 - p\cos x - q\sin x \rightarrow (eq3)$

Substitute $(eq1)$, $(eq2)$ and $(eq3)$ into $2y'' + 3y' + y = x^2 + 3\sin x$

$$2(2a_2 - p\cos x - q\sin x) + 3(2a_2 x + a_1 - p\sin x + q\cos x)$$
$$+(a_2 x^2 + a_1 x + a_0 + p\cos x + q\sin x) = x^2 + 3\sin x$$

Equate the coefficient :

x^2 :

$a_2 = 1$

x^1 :

$6a_2 + a_1 = 0$

$6(1) + a_1 = 0$

$a_1 = -6$

x^0 :

$4a_2 + 3a_1 + a_0 = 0$

$4(1) + 3(-6) + a_0 = 0$

$a_0 = 14$

$\sin x$:

$-2q - 3q + q = 3$

$-q - 3p = 3 \rightarrow (eq4)$

$\cos x$:

$-2p + 3q + p = 0$

$3q - p = 0$

$p = 3q \rightarrow (eq5)$

Substitute $(eq5)$ into $(eq4)$

$-q - 3(3q) = 3$

$-10q = 3$

$q = -\frac{3}{10}$

$p = 3(-\frac{3}{10})$

$p = -\frac{9}{10}$

$\therefore\ y = Ae^{-\frac{x}{2}} + Be^{-x} + x^2 - 6x + 14 - \frac{9}{10}\cos x - \frac{3}{10}\sin x$

Question 4

Solve the given initial problem

(a) $y'' + y' - 2y = 2x$; $y(0) = 0$; $y'(0) = 1$

<u>Solution:</u>

$m^2 + m - 2 = 0$

$(m + 2)(m - 1) = 0$

$m = -2$ or $m = 1$

Roots are real and different

$\therefore\ y_h = Ae^{-2x} + Be^{x}$

$f(x) = 2x$

$y_k = a_1 x + a_0 \rightarrow (eq1)$

$y'_k = a_1 \rightarrow (eq2)$

$y''_k = 0 \rightarrow (eq3)$

Substitute $(eq1)$, $(eq2)$ and $(eq3)$ into $y'' + y' - 2y = 2x$

$(0) + (a_1) - 2(a_1 x + a_0) = 2x$

Equate the coefficient :

x^1 :

$-2a_1 = 2$

$a_1 = -1$

x^0 :

$a_1 - 2a_0 = 0$

$(-1) - 2a_0 = 0$

$a_0 = -\frac{1}{2}$

$\therefore\ y = Ae^{-2x} + Be^{x} - x - \frac{1}{2}$

$y(0) = 0$:

$0 = Ae^{-2(0)} + Be^{(0)} - (0) - \frac{1}{2}$

$A + B = \frac{1}{2} \rightarrow (eq4)$

$y'(0) = 1$:

$y' = -2Ae^{-2x} + Be^{x} - 1$

$1 = -2Ae^{-2(0)} + Be^{(0)} - 1$

$2A - B = -2$

$B = -2 - 2A \rightarrow (eq5)$

Substitute ($eq5$) into ($eq4$)

$A + (-2 - 2A) = \frac{1}{2}$

$A = -\frac{1}{2}$

$B = -2 - 2(-\frac{1}{2})$

$B = 1$

$\therefore\ y = -\frac{1}{2}e^{-2x} + e^{x} - x - \frac{1}{2}$

(b) $y''+4y=3\sin 2x$; $y(0)=2$; $y'(0)=0$

Solution:

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$y_h = A\cos 2x + B\sin 2x$

$f(x)3\sin 2x$

$y_k = x(p\cos 2x + q\sin 2x) \rightarrow (eq1)$

$y'_k = p\cos 2x + q\sin 2x + x(-2p\sin 2x + 2q\cos 2x) \rightarrow (eq2)$

$y''_k = -2p\sin 2x + 2q\cos 2x + (-2p\sin 2x + 2q\cos 2x) + x(-4cos 2x - 4q\sin 2x) \rightarrow (eq3)$

Substitute $(eq1)$, $(eq2)$ and $(eq3)$ into $y'' + 4y = 3\sin 2x$

$-2p\sin 2x + 2q\cos 2x + (-2p\sin 2x + 2q\cos 2x) + x(-4cos 2x - 4q\sin 2x)$

$+4x(p\cos 2x + q\sin 2x = 3\sin 2x$

Equate the coefficient:

$\sin 2x$:

$-2p - 2p = 3$

$-4p = 3$

$p = -\frac{3}{4}$

$\cos 2x$:

$2q + 2q = 0$

$4q = 0$

$q = 0$

$\therefore\ y = A\cos 2x + B\sin 2x - \frac{3}{4}x\cos 2x$

$y(0) = 2$

$2 = A\cos 2(0) + B\sin 2(0) - \frac{3}{4}(0)\cos 2(0)$

$A = 2$

$y'(0) = -1$

$y' = -2A\sin 2x + 2B\cos 2x - \frac{3}{4}\cos 2x + \frac{3}{2}x\sin 2x$

$-1 = -2A\sin 2(0) + 2B\cos 2(0) - \frac{3}{4}\cos 2(0) + \frac{3}{2}(0)\sin 2(0)$

$2B = -\frac{1}{4}$

$B = -\frac{1}{8}$

$\therefore\ y = 2\cos 2x - \frac{1}{8}\sin 2x - \frac{3}{4}x\cos 2x$

(c) $y'' - 2y' - 3y = 3xe^{2x}$; $y(0) = 1$; $y'(-1) = 1$

<u>Solution:</u>

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3$ or $m = -1$

$\therefore\ y_h = Ae^{3x} + Be^{-x}$

$f(x) = 3xe^{2x}$

$y_k = (a_1 x + a_0)(e^{2x}) \rightarrow (eq1)$

$y'_k = a_1 e^{2x} + 2a_1 xe^{2x} + 2a_0 e^{2x} \rightarrow (eq2)$

$y''_k = 2a_1 e^{2x} + 2a_1 e^{2x} + 4a_1 xe^{2x} + 4a_0 e^{2x} \rightarrow (eq3)$

Substitute ($eq1$), ($eq2$) and ($eq3$) into $y'' - 2y' - 3y = 3xe^{2x}$

$(2a_1 e^{2x} + 2a_1 e^{2x} + 4a_1 xe^{2x} + 4a_0 e^{2x}) - 2(a_1 e^{2x} + 2a_1 xe^{2x} + 2a_0 e^{2x})$

$-3(a_1 xe^{2x} + a_0 e^{2x}) = 3xe^{2x}$

Equate the coefficient:

xe^{2x} :

$-3a_1 - 4a_1 + 4a_1 = 3$

$-3a_1 = 3$

$a_1 = -1$

e^{2x} :

$-3a_0 - 2a_1 - 4a_0 + 4a_1 + 4a_0 = 0$

$-3a_0 - 2(-1) - 4a_0 + 4(-1) + 4a_0 = 0$

$-3a_0 = 2$

$a_0 = -\frac{2}{3}$

$\therefore\ y = Ae^{3x} + Be^{-x} - xe^{2x} - \frac{2}{3}e^{2x}$

$y(0) = 1$

$1 = Ae^{3(0)} + Be^{-(0)} - (0)e^{2(0)} - \frac{2}{3}e^{2(0)}$

$A + B = \frac{5}{3} \rightarrow (eq4)$

$y'(0) = 0$

$y' = 3Ae^{3x} - Be^{-x} - 2xe^{2x} - \frac{4}{3}e^{2x} - e^{2x}$

$0 = 3Ae^{3(0)} - Be^{-(0)} - 2(0)e^{2(0)} - \frac{4}{3}e^{2(0)} - e^{2(0)}$

$0 = 3A - B - 1 - \frac{4}{3}$

$3A - B = \frac{7}{3}$

$B = 3A - \frac{7}{3} \rightarrow (eq5)$

Substitute ($eq5$) into ($eq4$)

$A + (3A - \frac{7}{3}) = \frac{5}{3}$

$A = 1$

$B = 3(1) - \frac{7}{3}$

$B = \frac{2}{3}$

$\therefore\ y = e^{3x} + \frac{2}{3}e^{-x} - xe^{2x} - \frac{2}{3}e^{2x}$

Question 5

Find the general solution of the given differential equation by using the method of variation of parameters.

(a) $y''+4y=3\csc 2x$

<u>Solution:</u>

Step 1:

$$m^2+4=0$$
$$m=\pm 2i$$
$$y_h=A\cos 2x+B\sin 2x$$
$$y_1=\cos 2x \ , \quad y_2=\sin 2x$$

Step 2:

$$w=\begin{vmatrix}\cos 2x & \sin 2x\\ -2\sin 2x & 2\cos 2x\end{vmatrix}$$
$$=2\cos^2 2x+2\sin^2 2x$$
$$=2$$

Step 3:

$$u=-\int\frac{(\sin 2x)(3\csc 2x)}{1(2)}dx$$
$$=-\frac{3}{2}x+c_1$$

$$v=\int\frac{(\cos 2x)(3\csc 2x)}{1(2)}dx$$
$$=\int\frac{3}{2}\frac{\cos 2x}{\sin 2x}dx$$
$$=\frac{3}{4}\ln|u|+c_2$$
$$=\frac{3}{4}\ln|\sin 2x|+c$$

Step 4:

$$y=(-\frac{3}{2}x+c_1)\cos 2x+(\frac{3}{4}\ln|\sin x|+c_2)\sin 2x$$

(b) $y''-5y'+6y=2e^{x}$

<u>Solution:</u>

Step 1:

$$m^2-5m+6=0$$
$$(m-2)(m-3)=0$$
$$m=2 \text{ or } m=3$$
$$y_h=Ae^{2x}+Be^{3x}$$
$$y_1=e^{2x}\ ,\ \ y_2=e^{3x}$$

Step 2:

$$w=\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$
$$=e^{5x}$$

Step 3:

$$u=-\int \frac{e^{3x}2e^{x}}{e^{5x}}dx$$
$$=2e^{-x}+c_1$$

$$v=\int \frac{e^{2x}2e^{x}}{e^{5x}}dx$$
$$=2\int e^{-2x}dx$$
$$=-e^{-2x}+c_2$$

Step 4:

$$y=(2e^{-x}+c_1)e^{2x}+(-e^{-2x}+c_2)e^{3x}$$

(c) $4y''-4y'+y=16e^{\frac{x}{2}}$

<u>Solution:</u>

Step 1:

$$4m^2-4m+1=0$$
$$(m-1)^2=0$$
$$m=\frac{1}{2}$$
$$y_h=Ae^{\frac{1}{2}x}+Bxe^{\frac{1}{2}x}$$
$$y_1=e^{\frac{1}{2}x}\ ,\ \ y_2=xe^{\frac{1}{2}x}$$

Step 2:

$$w = \begin{vmatrix} e^{\frac{1}{2}x} & xe^{\frac{1}{2}x} \\ \frac{1}{2}e^{\frac{1}{2}x} & e^{\frac{1}{2}x} + \frac{1}{2}xe^{\frac{1}{2}x} \end{vmatrix}$$

$$= e^{x}$$

Step 3:

$$u = -\int \frac{xe^{\frac{1}{2}x}16e^{\frac{1}{2}x}}{4e^{x}}dx$$

$$= -2x^2 + c_1$$

$$v = \int \frac{e^{\frac{1}{2}x}16e^{\frac{1}{2}x}}{4e^{x}}dx$$

$$= 2\int e^{-2x}dx$$

$$= 4x + c_2$$

Step 4:

$$y = (-2x^2 + c_1)e^{\frac{1}{2}x} + (4x + c_2)xe^{\frac{1}{2}x}$$

Question 6

Solve the given initial value problem by using the method of variation of parameters.

(a) $y'' - 4y = e^{2x}$; $y(0) = 1$; $y'(0) = 0$

Solution:

Step 1:

$$m^2 - 4 = 0$$

$$m = \pm\frac{1}{2}$$

$$y_h = Ae^{2x} + Be^{-2x}$$

$$y_1 = e^{2x}, \quad y_2 = e^{-2x}$$

Step 2:

$$w = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$

$$= -4$$

Step 3:

$$u = -\int \frac{e^{2x} e^{2x}}{1(-4)} dx$$

$$= \frac{1}{4} x + c_1$$

$$v = \int \frac{e^{2x} e^{2x}}{1(-4)} dx$$

$$= -\frac{1}{16} e^{4x} + c_2$$

Step 4:

$$y = (\frac{1}{4} x + c_1) e^{2x} + (-\frac{1}{16} e^{4x} + c_2) e^{-2x}$$

$y(0) = 1$:

$$1 = c_1 + (-\frac{1}{16} + c_2)$$

$$c_1 + c_2 = \frac{17}{16} \rightarrow (eq1)$$

$y'(0) = 0$:

$$y' = \frac{1}{4} e^{2x} + 2e^{2x} (\frac{1}{4} x + c_1) - \frac{1}{4} e^{4x} (e^{2x}) - 2e^{2x} (-\frac{1}{16} e^{4x} + c_2)$$

$$0 = \frac{1}{4} + 2c_1 - \frac{1}{4} - 2(-\frac{1}{16} + c_2)$$

$$c_1 - c_2 = -\frac{1}{16} \rightarrow (eq2)$$

$\mathbf{(}eq1\mathbf{)} + \mathbf{(}eq2\mathbf{)}$;

$$2c_1 = 1$$

$$c_1 = \frac{1}{2}$$

$$c_2 = \frac{17}{16} - \frac{1}{2} = \frac{9}{16}$$

$$\therefore y = (\frac{1}{4} x + \frac{1}{2}) + (-\frac{1}{16} e^{4x} + \frac{9}{16}) e^{-2x}$$

(b) $4y''+y=2\sec(\frac{x}{2})$

<u>Solution:</u>

Step 1:

$$4y''+y=2\sec(\frac{x}{2})$$

$$4m^2+1=0$$

$$m=\pm\frac{1}{2}i$$

$$y_h=A\cos(\frac{x}{2})+B\sin(\frac{x}{2})$$

$$y_1=\cos(\frac{x}{2}),\ y_2=\sin(\frac{x}{2})$$

Step 2:

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$=\begin{vmatrix} \cos(\frac{x}{2}) & \sin(\frac{x}{2}) \\ -\frac{1}{2}\sin(\frac{x}{2}) & \frac{1}{2}\cos(\frac{x}{2}) \end{vmatrix}$$

$$=\frac{1}{2}\cos^2(\frac{x}{2})+\frac{1}{2}\sin^2(\frac{x}{2})$$

$$=\frac{1}{2}(1)$$

$$=\frac{1}{2}$$

Step3:

$$u=-\int\frac{(\sin(\frac{x}{2}))(2\sec(\frac{x}{2}))}{4(\frac{1}{2})}dx$$

$$=-\int\frac{\sin(\frac{x}{2})}{\cos(\frac{x}{2})}dx$$

$$=-\int\frac{\sin(\frac{x}{2})}{u}\cdot\frac{-2du}{\sin(\frac{x}{2})}$$

$$= -\int \frac{2}{u} du$$
$$= 2\ln|u| + c_1$$
$$= 2\ln\left|\cos(\frac{1}{2}x)\right| + c_1$$

$$v = \int \frac{(\cos(\frac{x}{2}))(2\sec(\frac{x}{2}))}{4(\frac{1}{2})} dx$$
$$= -\int 1 \quad dx$$
$$= x + c_2$$

Step 4:

$$y = uy_1 + vy_2$$
$$= (2\ln\left|\cos(\frac{x}{2})\right| + c_1)(\cos(\frac{x}{2})) + (x + c_2)(\sin\frac{x}{2})$$
$$= c_1\cos(\frac{x}{2}) + c_2\sin(\frac{x}{2}) + 2\ln\cos(\frac{x}{2})\cos(\frac{x}{2}) + x\sin(\frac{x}{2})$$

(c) $y'' - y' - 2y = 2e^{-x}$

<u>Solution:</u>

Step 1:

$$y'' - y' - 2y = 2e^{-x}$$
$$m^2 - m - 2 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2 \text{ or } m = -1$$
$$y_h = Ae^{2x} + Be^{-x}$$
$$y_1 = e^{2x}, \quad y_2 = e^{-x}$$

Step 2:

$$W = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$$
$$= -e^{-x} - 2e^{x}$$
$$= -3e^{x}$$

Step 3:

$$u = \int \frac{-e^{-x} - 2e^{x}}{1(-3e^{x})} dx$$

$$= \frac{2}{3} \int e^{-3x} dx$$

$$= -\frac{2}{9} e^{-3x} + c_1$$

$$v = \int \frac{e^{2x} 2e^{-x}}{1(-3e^{x})} dx$$

$$= -\frac{2}{3} x + c_2$$

Step 4:

$$y = uy_1 + vy_2$$

$$= (-\frac{2}{9} e^{-3x} + c_1) e^{2x} + (-\frac{2}{3} x + c_2) e^{-x}$$