CHAPTER TWO

COMPLEX NUMBERS

Introduction

In this chapter, we will work with imaginary numbers. Up until now, you've been told that you can't take the square root of a negative number. That's because you had no numbers that, when squared, were negative. Every number was positive after you squared it. So you couldn't very well square-root a negative and expect to come up with anything sensible. Now, however, you can take the square root of a negative number, but it involves using a new number to do it. Anyway, this new number is called "i", standing for "imaginary", because "everybody knew" that i wasn't "real". The imaginary is defined to be $i = \sqrt{-1}$.

Objectives

After completing these tutorials, students should be able to:

- Simplify the given complex number expressions.
- Find the conjugate of complex number.
- Express the given fractions in the form $a + bi$ and represent each on the Argand diagram.
- Solve the given complex equations.
- Find the modulus and argument of the given complex numbers
- Express the complex number in polar form using its principal argument in radiant units.
- Use De Moivre's Theorem to simplify complex number expressions.
- Express the complex number in the exponential form (Euler's formula).

Question 1

Simplify

(a) $(3+4i)(3-4i)$

Solution:

$(3+4i)(3-4i)$

$=9-12i+12i-16i^2$

$=9-16(-1)$

$=25$

(b) $(a+bi)^2$

Solution:

$(a+bi)^2$

$=a^2+2abi+b^2i^2$

$=a^2+2abi+b^2(-1)$

$=a^2-b^2+2abi$

(c) $i(1+i)(2+i)$

Solution:

$i(1+i)(2+i)$

$=i(2+i+2i+i^2)$

$=i(2+3i-1)$

$=i(1+3i)$

$=i+3i^2$

$=i-3$

Question 2

Find the conjugate of z.

(a) $z=-i$

Solution:

$\overline{z}=i$

(b) $z=10+2i$

Solution:

$\overline{z}=10-2i$

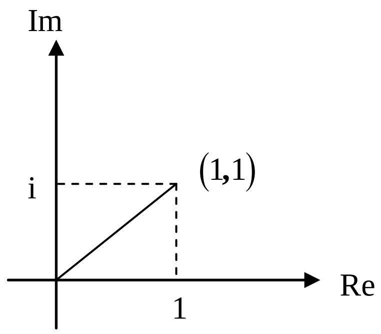
Question 3

Express the following fractions in the form $a+bi$ and represent each on the Argand diagram.

(a) $\dfrac{2}{1-i}$

Solution:

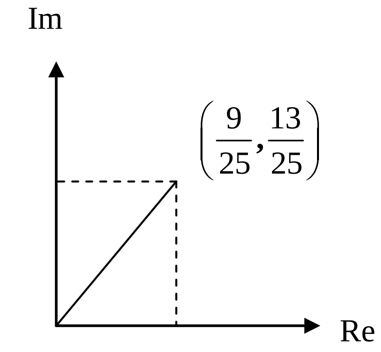
$$\frac{2}{1-i}$$
$$=\frac{2}{1-i}\times\frac{1+i}{1+i}$$
$$=\frac{2(1+i)}{1^2-i^2}$$
$$=\frac{2(1+i)}{1+1}$$
$$=\frac{2(1+i)}{2}$$
$$=1+i$$



(b) $\dfrac{3+i}{4-3i}$

Solution:

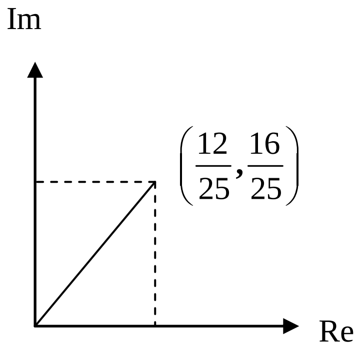
$$\frac{3+i}{4-3i}$$
$$=\frac{3+i}{4-3i}\times\frac{4+3i}{4+3i}$$
$$=\frac{(3+i)(4+3i)}{16-9i^2}$$
$$=\frac{12+9i+4i+3i^2}{16-9i^2}$$
$$=\frac{12+13i-3}{16+9}$$
$$=\frac{9+13i}{25}$$
$$=\frac{9}{25}+\frac{13}{25}i$$



(c) $\dfrac{4i}{4+3i}$

Solution:

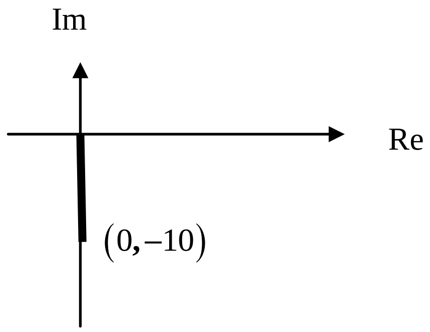
$$\frac{4i}{4+3i}$$
$$=\frac{4i}{4+3i}\times\frac{4-3i}{4-3i}$$
$$=\frac{4i(4-3i)}{16-9i^2}$$
$$=\frac{16i-12i^2}{16+9}$$
$$=\frac{12+16i}{25}$$
$$=\frac{12}{25}+\frac{16}{25}i$$



(d) $\dfrac{10}{i}$

Solution:

$$\frac{10}{i}$$
$$=\frac{10}{i}\times\frac{-i}{-i}$$
$$=\frac{-10i}{-i^2}$$
$$=-10i$$



Question 4

Solve the following equations for x and y.

(a) $x+yi=(3+i)(2-3i)$

Solution:

$$\begin{aligned}x+yi&=(3+i)(2-3i)\\&=6-9i+2i-3i^2\\&=6-7i-3(-1)\\&=9-7i\end{aligned}$$
$\therefore x=9, y=-7$

(b) $3+4i=(x+yi)(1+i)$

<u>Solution:</u>

$$3+4i=(x+yi)(1+i)$$
$$=x+xi+yi+yi^2$$
$$=x-y+(x+y)i$$
$$\Rightarrow x-y=3 \quad ---(I)$$
$$x+y=4 \quad ---(II)$$
$$(I)+(II)$$
$$2x=7$$
$$x=\frac{7}{2}$$

Substituted

$x=\frac{7}{2}$ into (II)

$$\frac{7}{2}+y=4$$
$$y=\frac{1}{2}$$
$$\therefore x=\frac{7}{2}, y=\frac{1}{2}$$

(c) $\dfrac{2+5i}{1-i} = x + yi$

Solution:

$$\frac{2+5i}{1-i} = x + yi$$

$$\begin{aligned} x + yi &= \frac{2+5i}{1-i} \\ &= \frac{2+5i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{(2+5i)(1+i)}{1-i^2} \\ &= \frac{2+2i+5i+5i^2}{2} \\ &= \frac{2+7i-5}{2} \\ &= \frac{-3+7i}{2} \\ &= \frac{-3}{2} + \frac{7}{2}i \end{aligned}$$

$$\therefore x = -\frac{3}{2}, y = \frac{7}{2}$$

(d) $x + yi = 2$

Solution:

$$\begin{aligned} x + yi &= 2 \\ &= 2 + 0i \end{aligned}$$

$$\therefore \quad x = 2, \ y = 0$$

Question 5

Find the modulus and argument of the following complex numbers

(a) $(1-i)(4+3i)$

Solution:

$$z=(1-i)(4+3i)$$
$$=4+3i-4i-3i^2$$
$$=4-i-3(-1)$$
$$=7-i$$
$$|z|=\sqrt{7^2+(-1)^2}$$
$$=\sqrt{49+1}$$
$$=\sqrt{50}$$
$$q=\tan^{-1}(\frac{-1}{7})$$
$$=-0.14$$

(b) $\frac{1+7i}{1+i}$

Solution:

$$z=\frac{1+7i}{1+i}$$
$$=\frac{1+7i}{1+i}\times\frac{1-i}{1-i}$$
$$=\frac{(1+7i)(1-i)}{1-i^2}$$
$$=\frac{1-i+7i-7i^2}{2}$$
$$=\frac{1+6i-7(-1)}{2}$$
$$=\frac{8+6i}{2}$$
$$=4+3i$$

$$|z|=\sqrt{4^2+3^2}$$
$$=\sqrt{16+9}$$
$$=\sqrt{25}$$
$$=5$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$
$$= 0.64$$

Question 6

Given $z_1 = -2 + i$ and $z_2 = 1 - i$. Find $|z_1 z_2|$.

Solution:

$z_1 = -2 + i$

$|z_1| = \sqrt{(-2)^2 + 1^2}$

$= \sqrt{5}$

$z_2 = 1 - i$

$|z_2| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$

$|z_1 z_2|$

$= |z_1||z_2|$

$= \sqrt{5} \times \sqrt{2}$

$= \sqrt{10}$

Question 7

Express the complex number in polar form using its principal argument in radiant units.

(a) $z = -3 + 2i$

Solution:

$z = -3 + 2i$

$r = |z|$

$= \sqrt{(-3)^2 + 2^2}$

$= \sqrt{13}$

$$\theta = \tan^{-1}\left(\frac{2}{-3}\right) = 2.554$$

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{13}\left(\cos 2.554 + i\sin 2.554\right)$

(b) $z = 2\sqrt{3} - \sqrt{3}i$

Solution:

$z = 2\sqrt{3} - \sqrt{3}i$

$$
\begin{aligned}
r &= |z| \\
&= \sqrt{(2\sqrt{3})^2 + (-\sqrt{3})^2} \\
&= \sqrt{12 + 3} \\
&= \sqrt{15}
\end{aligned}
$$

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{-\sqrt{3}}{2\sqrt{3}}\right) \\
&= -\tan^{-1}\left(\frac{\sqrt{3}}{2\sqrt{3}}\right) \\
&= -\tan^{-1}\left(\frac{1}{2}\right) \\
&= -0.464
\end{aligned}
$$

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta) \\
&= \sqrt{15}\left(\cos(-0.464) + i\sin(-0.464)\right)
\end{aligned}
$$

Question 8

By using De Moivre's Theorem, find $(2+2i)^{\frac{1}{3}}$. Give your answer in polar form.

Solution:

Let

$z = 2 + 2i$

$$
\begin{aligned}
|z| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{8} \\
&= 2\sqrt{2}
\end{aligned}
$$

$$\theta = \mathbf{tan}^{-1}(\frac{2}{2})$$

$$= \mathbf{tan}^{-1}(1)$$

$$= \frac{\pi}{4}$$

De Moivre's Theorem

$$z^t = (x + yi)^t = r(\mathbf{cos}\, t\theta + i\, \mathbf{sin}\, t\theta)$$

$$\Rightarrow (2 + 2i)^{\frac{1}{3}} = r(\mathbf{cos}\, \frac{1}{3}\theta + i\, \mathbf{sin}\, \frac{1}{3}\theta)$$

$$= 2\sqrt{2}\left(\mathbf{cos}(\frac{1}{3} \times \frac{\pi}{4}) + i\, \mathbf{sin}(\frac{1}{3} \times \frac{\pi}{4}) \right)$$

$$= 2\sqrt{2}\left(\mathbf{cos}\, \frac{\pi}{12} + i\, \mathbf{sin}\, \frac{\pi}{12} \right)$$

Question 9

By using De Moivre's Theorem find

(a) $(1+i)^5$

Solution:

Let

$z = 1 + i$

$$|z| = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\theta = \mathbf{tan}^{-1}\left(\frac{1}{1}\right)$$

$$= \mathbf{tan}^{-1}(1)$$

$$= \frac{\pi}{4}$$

De Moivre's Theorem

$$z^t = (x + yi)^t = r(\mathbf{cos}\, t\theta + i\, \mathbf{sin}\, t\theta)$$

$$\Rightarrow (1+i)^5 = r\left(\cos 5\theta + i\sin 5\theta\right)$$

$$= \sqrt{2}\left(\cos(5\times\frac{\pi}{4}) + i\sin(5\times\frac{\pi}{4})\right)$$

$$= \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

(b) $(\sqrt{3}-i)^9$

<u>Solution:</u>

Let

$z = \sqrt{3} - i$

$$|z| = \sqrt{3^2 + (-1)^2}$$

$$= \sqrt{10}$$

$$\theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$

$$= -\frac{\pi}{6}$$

De Moivre's Theorem

$$z^t = (x+yi)^t = r(\cos t\theta + i\sin t\theta)$$

$$\Rightarrow (\sqrt{3}-i)^9 = r\left(\cos 9\theta + i\sin 9\theta\right)$$

$$= \sqrt{10}\left(\cos(9\times\frac{-\pi}{6}) + i\sin(9\times\frac{-\pi}{6})\right)$$

$$= \sqrt{10}\left(\cos(\frac{-3\pi}{2}) + i\sin(\frac{-3\pi}{2})\right)$$

Question 10

Express the complex number in the exponential form (Euler's formula).

(a) $2+i$

<u>Solution:</u>

Let

$z = 2 + i$

$$|z| = \sqrt{2^2 + 1^2}$$
$$= \sqrt{5}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$$

$$z = re^{\theta i}$$
$$= \sqrt{5}e^{0.464i}$$

(b) $-4-4i$

<u>Solution:</u>

Let

$z = -4 - 4i$

$$|z| = \sqrt{(-4)^2 + (-4)^2}$$
$$= \sqrt{16+16}$$
$$= \sqrt{32}$$
$$= 4\sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-4}{-4}\right)$$
$$= -\frac{3\pi}{4}$$

$$z = re^{i\theta}$$
$$= 4\sqrt{2}e^{-\frac{3\pi}{4}i}$$