# CHAPTER 2 DIFFERENTIATION

## 2.1 THE GEOMETRICAL MEANING OF DIFFERENTIATION (page 54)

**Definition 2.1 (The Derivative) (page 54)**

Let $y = f(x)$ is a function. The derivative of a function $f$ with respect to $x$, represented by $f'$, is defined by

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

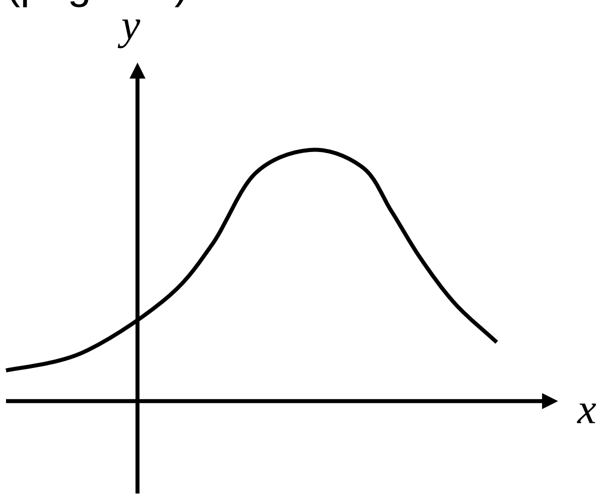
provided the limit exists.

The process to get the derivative using Definition 2.1 is called the differentiation by using the ***first principal***.

Notations for writing the derivative:

$$y',\ f' \qquad y'(x),\ f'(x) \qquad \dot{y},\ \dot{f} \qquad \frac{dy}{dx}, \frac{df}{dx}$$

Geometry to illustrate the concept of differentiation (page 55)




curve: $y = f(x)$;

points: $P(x_0, y_0)$ and $Q(x_1, y_1)$

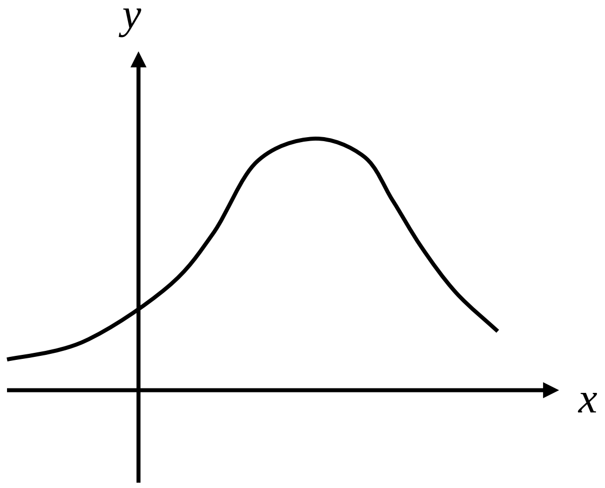
Gradient of the line $PQ$,

$$m_{PQ} = \frac{y_1 - y_0}{x_1 - x_0}$$

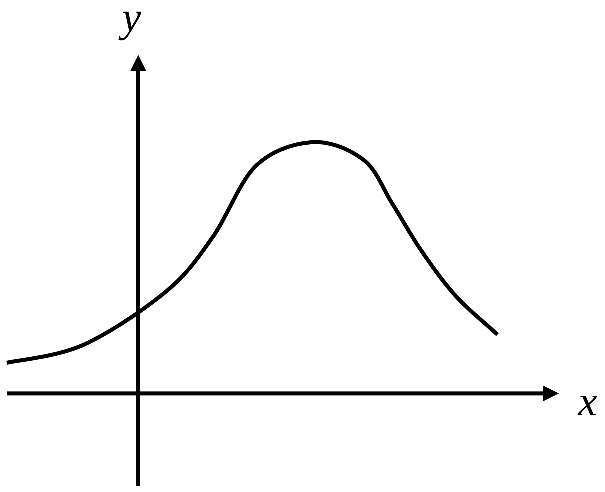
$$m_{PQ} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \text{eq.(2.2)}$$

$\delta x = x_1 - x_0 \Rightarrow x_1 = x_0 + \delta x$

then, $$m_{PQ} = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$




When $Q$ approaches $P$, $x_1$ will approaches $x_0$, hence $\delta x = x_1 - x_0$ will tend to zero.




When $Q$ is close enough to $P$, the straight line joining $P$ and $Q$ will become tangent ot the curve at $P$.

Therefore, the gradient of the tangent at $P$ is

$$m = \lim_{\delta x \to 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

Definition 2.2 (page 56)

Steps that involved in obtaining the derivatives of the functions **using definition** (differentiation from the **first principle**)

*__Step 1__*

Given $y = f(x)$. Write the expression $f(x + \delta x)$.

*__Step 2__*

Obtain the difference $f(x + \delta x) - f(x)$.

*__Step 3__*

Simplify the expression $\dfrac{f(x + \delta x) - f(x)}{\delta x}$.

*__Step 4__*

Find the limit

$$f'(x) = \operatorname*{had}_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

***Example 2.1(page 57):***

By using the differentiation from the first principal, find the derivatives of the following functions:

(a) $y = 9$ (b) $y = 2x^2$ (d) $y = \dfrac{1}{x}$

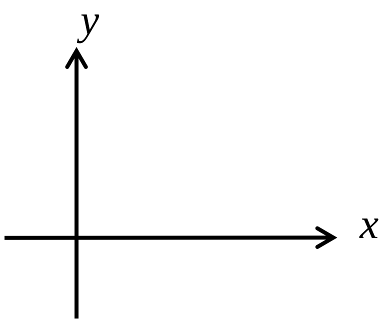
***Exercise:***

$f(x) = 3x^2 + 5$

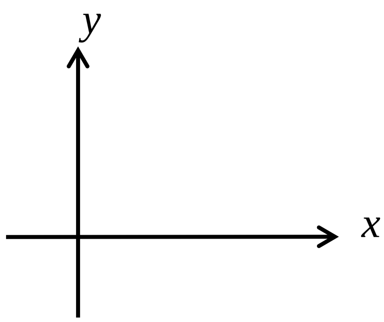
## Remarks (page 60-61):

A function is not differentiable when its graph has

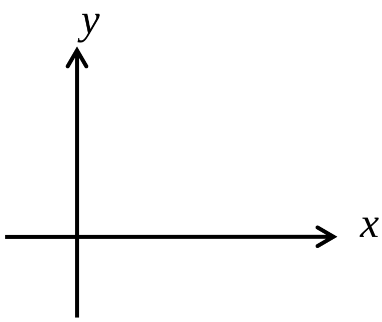
(i) points of discontinuity




(ii) vertices




(iii) vertical tangent

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**Theorem 2.1 The Relation between Continuity and Differentiable (page 64)**

If $f$ is differentiable at $x = c$, then $f$ is continuous at $x = c$.

## Remarks (page 64):

A function which is continuous at a point may be not differentiable at that point.

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## 2.2 DIFFERENTIATION OF SIMPLE ALGEBRAIC FUNCTION (page 65)

**Theorem 2.2 (Derivative for Constant Function) (page 65)**

If $y = c$, $c$ is a constant, for all $x$, then

$$\frac{dy}{dx} = 0.$$

***Example 2.5 (page 66):***

Differentiate the following functions with respect to $x$

(a) $y = \frac{100}{3}$ (b) $y = \pi$

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**Teorem 2.3 (The Derivative for Positive Power Integer)**

If $y = x^n$, and $n$ is a positive integer, then

$$\frac{dy}{dx} = nx^{n-1}$$

***Example 2.6 (page 67):***

Find the derivatives of the following functions.

(a) $y = x^{15}$ (b) $f(x) = x^{99}$

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## 2.3 DIFFERENTIATION RULES (page 68)

**Theorem 2.4 (Differentiation of Multiples)**

Let $y = cu$. If $u$ si a differentiable function of $x$, and $c$ is a constant, then

$$\frac{dy}{dx} = c\frac{du}{dx}$$

***Example 2.7 (page 68):***

Find the derivatives of the following functions.

(a) $y = 4x^9$ (b) $y = -x^{15}$

**Theorem 2.5 (Defferentiation of Sums) (page 69)**

Let $u$ and $v$ be differentiable functions of $x$. If $y = u + v$, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

**Remarks: (Defferentiation of Differences) (page 69)**

**If** $y = u - v$ **then** $$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

***Example 2.8 & 2.9 (page 69-70):***

Differentiate the following functions with respect to $x$

(a) $y = 4x^2 + 5x^3$ (b) $y = (2x^2 - x)^2$

**Theorem 2.6 (Differentiation of Products) (page 70)**

Let $u$ and $v$ are differentiable functions with respect to $x$.

If $y = uv$, then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

***Example 2.10 (page 71):***

Differentiate $y$ with respect to $x$ if

$y = (4x^2 + 2)(3x^3 - 1)$

**Remarks (page 72):**

In some cases, it is easier to expand the expression first before finding its derivative. However, in cases when expansion is not possible or practical, the rule on differentiation of products will be needed.

**Theorem 2.7 (Differentiation of Quotient) (page 72)**

Let *u* and $v \neq 0$ be differentiable functions with respect to *x*. If $y = \dfrac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

***Example 2.11 & 2.12 (page 73 & 74):***

Differentiate the following functions with respect to $x$

(a) $y = \dfrac{7 - 3x^2}{3 - x}$ (b) $y = \dfrac{1}{x^3 - 2x + 5}$

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**Theorem 2.8 (The Power Rule for All Integer) (page 75)**

If $y = x^n$, and $n$ is any integer, then

$$\frac{dy}{dx} = nx^{n-1}$$

**Notes:** Theorem 2.3 is for positive power integer only while Theorem 2.8 is an extension for all integer $n$.

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For $n = 0$, (page 75)

$$\frac{dy}{dx} = \frac{d}{dx}[1] = 0x^{n-1} = 0$$

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***Example 2.13 (page 75):***

Differentiate the following functions with respect to $x$

(a) $f(x) = x^{-9}$ (b) $y = \dfrac{1}{x}$

***Exercise:***

Find the derivatives of the following functions.

(a) $y = a^3$, $a$ is a constant (b) $y = ax^n$, $a$ is a constant

(c) $f(x) = \frac{1}{3}x^5$ (d) $y = 0.5x^2$ (e) $f(x) = \sqrt{x}$

(f) $f(x) = \frac{x^3 - 2x + 3}{x^2 + 7x - 1}$ (g) $y = (2 - x^3)(x^3 + x - 1)$

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**Summary (page 76):**

The table below shows all the differentiation rules where $u$ and $v$ are differentiable functions with respect to $x$ and $c$ is a constant.

| $f(x)$ | $f'(x)$ |
|---|---|
| $y = cu$ | $\frac{dy}{dx} = c\frac{du}{dx}$ |
| $y = u \pm v$ | $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$ |
| $y = uv$ | $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ |
| $y = \frac{u}{v}$ | $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ |

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## 2.4 HIGHER ORDER DIFFERENTIATIONS

Differentiation of second order or higher. (page 78)

$$f'(x) = \frac{d}{dx}\left[f(x)\right]$$ **First differential coefficient**

$$f''(x) = (f')' = \frac{d}{dx}\left[\frac{d}{dx}[f(x)\right] = \frac{d^2}{dx^2}\left[f(x)\right]$$ **Second differential coefficient**

$$f'''(x) = (f'')' = \frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right]\right] = \frac{d^3}{dx^3}\left[f(x)\right]$$

$$f^{(4)}(x) = (f''')' = \frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}\left[\frac{d}{dx}[f(x)]\right]\right]\right] = \frac{d^4}{dx^4}\left[f(x)\right]$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$f^{(n)}(x) = (f^{(n-1)})' = \frac{d^n}{dx^n}[f(x)]$$ ***n*th differential coefficient**

(We use the prime symbol only for differentiation up to order three. For derivative of higher order we use integers brackets to represent the ***orders*** of differentiation.)

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1. If $y = f(x)$, then the sequence of differentiations can also be written as

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ \ldots, \quad \frac{d^ny}{dx^n}, \ \ldots$$

or can be simplified as

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ \ldots, \quad y^{(n)}, \ \ldots$$

2. The notation represents the values of order of differentiation at certain value $x_0$

$$y''(x_0), \quad y^{(5)}(x_0), \quad \left.\frac{d^2y}{dx^2}\right|_{x=x_0}, \quad \left.\frac{d^5y}{dx^5}\right|_{x=x_0}$$

***Example 2.14 (page 79):***

Show that $y = x^3 + 3x + 1$ satisfied the equation

$y''' + xy'' - 2y' = 0$.

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***Example 2.15 (page 79):***

If $y = x^5 + 3x^3 - 2x + 1$, find the values of $\left.\frac{d^3y}{dx^3}\right|_{x=2}$ and $y^{(4)}(2)$.

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***Exercise at home: (Tutorial 3)***

(page 65) Quiz 2A: No. 1a), 1b), 2a)

(page 67) Quiz 2B: No. 1, 2

(page 76) Quiz 2C: No. 1a), 1c), 1e), 2a), 2c), 2e), 3a), 3b)

(page 81) Quiz 2D: No. 1c), 3a), 3b), 4a)

(page 141) Exercise 2: No. 1b), 1d), 6a)

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## 2.5 THE CHAIN RULE

**Theorem 2.9 (The Chain Rule) (page 83)**

If $g$ is differentiable at $x$ and $f$ is differentiable at the point $g(x)$, then the composite function $f \circ g$ is differentiable at $x$. In other word, if $y = f(g(x))$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Two special formulas using the chain rule when $u$ is differentiable function with respect to $x$.

| $u(x)$ | $u'(x)$ |
|---|---|
| $u^{1/2}$ | $\frac{1}{2u^{1/2}}\frac{du}{dx}$ (*) |
| $u^n$ | $nu^{n-1}\frac{du}{dx}$ (**) |

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***Example 2.19 (page 84):***

Find $\frac{dy}{dx}$ for the following functions.

a) $y=\left(x^3+6x-3\right)^{20}$ b) $y=\sqrt{3-2x}$

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Sometime the chain rule involves more than two functions.

For example (page 87 below)

(a) $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$

(b) $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dw}\cdot\frac{dw}{ds}\cdot\frac{ds}{dt}\cdot\frac{dt}{dx}$

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***Example 2.22 (page 88):***

Find $\frac{dy}{dx}$ if $y=\sqrt{4+3\sqrt{x}}$ .

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***Exercise:***

1. If $f(x)=3x^4-2x^3+x^2-4x+2$, find its first five derivatives.

2. Find $\frac{dy}{dx}$ for the following functions.

a) $y = \frac{1}{(1-2x)^3}$ b) $y = (2x-1)^3(x+1)^5$

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## 2.6 DIFFERENTIATION FO TRIGONOMETRIC FUNCTIONS

| Differentiation Formulas for the Trigonometric Functions (page 90, 93, 97) | | Differentiation Formulas for the Trigonometric Functions when $u = g(x)$ is a function of $x$ (page 100) |
|---|---|---|
| $\frac{d}{dx}[\sin x] = \cos x$ | | $\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$ |
| $\frac{d}{dx}[\cos x] = -\sin x$ | | $\frac{d}{dx}[\cos u] = -\sin u \frac{du}{dx}$ |
| $\frac{d}{dx}[\tan x] = \sec^2 x$ | | $\frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx}$ |
| $\frac{d}{dx}[\text{cosec } x] = -\text{cosec } x \cot x$ | | $\frac{d}{dx}[\text{cosec } u] = -\text{cosec } u \cot u \frac{du}{dx}$ |
| $\frac{d}{dx}[\sec x] = \sec x \tan x$ | | $\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx}$ |
| $\frac{d}{dx}[\cot x] = -\text{cosec}^2 x$ | | $\frac{d}{dx}[\cot u] = -\text{cosec}^2 u \frac{du}{dx}$ |

***Example 2.23 (page 91):***

(b) $y = 5x\sin x$

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***Example 2.24 (page 91):***

(a) $y = \sin(2x-5)$ (b) $y = \sin^3(2x-5)$

***Example 2.26 (page 95):***

(d) $y = \dfrac{\cos x}{1 - \cos x}$

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***Example 2.29 (page 97):***

(a) $y = 4\tan\left(\dfrac{x}{2}\right)$

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***Example 2.31 (page 99)*:**

(a) Find $\dfrac{d^2y}{dx^2}$ if

(i) $y = x + \tan x$

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## 2.7 DIFFERENTIATION OF LOGARITHMIC FUNCTION

(page 103 & 104)

| | | |
|---|---|---|
| If $y = \log_a x$, then $$\frac{dy}{dx} = \frac{1}{x}\log_a e,$$ or $$\frac{d}{dx}[\log_a x] = \frac{1}{x}\log_a e, x > 0$$ | | In general, if $u = u(x)$ and $y = \log_a u$, then by using the chain rule we have $$\frac{d}{dx}[\log_a u] = \frac{1}{u}\log_a e\frac{du}{dx}$$ |
| In the special case $a = e$, we have $$y = \log_e x = \ln x$$ therefore | | Further more when $a = e$ $$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}, \ u > 0$$ |

| | | |
|---|---|---|
| $\frac{dy}{dx} = \frac{1}{x} \ln e$<br>Since $\ln e = 1$<br>$\boxed{\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0}$ | | |

***Example 2.33 (page 104):***

(a) $y = \log_2 x^5$

***Example 2.34 (page 105):***

(b) $y = \ln(\sin x - \cos 2x)$ (c) $y = \ln \sqrt{\frac{1+2x}{1-2x}}$ (d) $y = \ln \sin^2 x$

## 2.8 DIFFERENTIATION OF EXPONENTIAL FUNCTION

Exponential functions $y = a^x$ and $y = e^x$

| Differentiation Formulas for the Exponential Functions (page 109) | | Differentiation Formulas for the Exponential Functions when $u = g(x)$ is a function of $x$ (page 109) |
|---|---|---|
| $\frac{d}{dx}\left[a^x\right] = a^x \ln a$ | | $\frac{d}{dx}\left[a^u\right] = a^u \ln a \frac{du}{dx}$ |
| $\frac{d}{dx}\left[e^x\right] = e^x$ | | $\frac{d}{dx}\left[e^u\right] = e^{u(x)} \frac{du}{dx}$ |

***Example 2.37 (page 109):***

(a) $y = 10^x$ (b) $y = 10^{2x-5}$

***Example 2.38 (page 111):***

(a) $y = e^{\frac{1}{2}x}$ (c) $y = e^{2x^2}$

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***Example 2.40 (page 112):***

If $y = e^{-x}\cos 3x$, find

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$

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## 2.9 DIFFERENTIATION OF IMPLICIT FUNCTIONS

In previous section we have discussed the differentiation of the function where $y$ can be expressed in terms of $x$, $y = f(x)$. This is call explicit definition. However, not all equations can be explicitly defined (where $y$ cannot be expressed in terms of $x$ only). Such a relation is said to be implicit defined and we write in the form of $F(x, y) = 0$.

The method to obtain differential coefficients of functions, which are implicitly defined, is called ***implicit differentiation***.

**Theorem 2.10 (Differentiation)**

If $y$ and $x$ are related in a function, and explicitly and implicitly defined, then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

***Example 2.41 (page 117):***

If $3x^2 - xy + 3y = 7$, find $\frac{dy}{dx}$,

a) by written $y$ in terms of $x$.

b) by using implicit differentiation.

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***Example 2.43 (Page 118):***

If $x^2 \sin y + 2x = y^2$, find $\frac{dy}{dx}$ in terms of $x$ and $y$.

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***Example 2.48 (page 123):***

If $y = \frac{A\cos 2x + B\sin 2x}{x}$, where $A$ and $B$ are constants,

prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0$

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***Example 2.49 (page 124):***

Find $\frac{dy}{dx}$ for the function

(b) $y = x^x$

## 2.10 DIFFERENTIATION OF PARAMETRIC FUCNTIONS (page 128)

In some cases, implicit functions can be expressed in terms of parameters. The implicit relationship of $x$ and $y$ can be expressed in a simpler form by using a third variable known as the **parameter**.

For example $x = t^2$ and $y = t^3 + t$

is a parametric equation of a curve with parameter $t$

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**(page 129):**

The differentiation of parametric function is an application of the chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \qquad \text{and also} \qquad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

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***Example 2.53 (page 129):***

A curve is given by a parametric equation

$$x = 2t \qquad \text{and} \qquad y = 4 - 4t - 4t^2$$

Find

(a) $\frac{dy}{dx}$ by using parametric differentiation

(b) $y$ in terms of $x$ and hence find $\frac{dy}{dx}$.

***Example 2.55 (page 130):***

The parametric equations of a curve are given by

$x = e^t$ and $y = \sin t$.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of $t$.

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***Exercise:***

Find $\dfrac{dy}{dx}$ for the given value of $t$.

(a) $x = t^2 - 3t + 1$ $\quad y = t^3 - 2t - 7$ $\quad t = 4$

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***Exercise at home:***

**Do Quiz 2E, 2F, 2G, 2H, 2I, 2J**

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***Exercise at home (Tutorial 4):***

***(page 141-150) Exercise 2:***

**Do no. 7a), 8a), 8d), 8f), 9b), 9c), 9d), 9e), 9i), 10a), 10b), 10c), 10e), 10f), 11a), 11e), 11g), 11h), 11k), 11l), 11m), 12c), 12f), 13a), 13b), 13c), 13e), 13i), 13o), 17b), 36b), 36c), 39a), 39b)**