CHAPTER TWO

LAPLACE TRANSFORM

After completing these tutorials, students should be able to:

- find the Laplace transform of the given function by using the definition of Laplace Transform
- find the Laplace transform of the given function by using the table of Laplace Transform
- sketch the graph of the given step function, and write the function in terms of unit step function
- find the Laplace transform of the given function by using the theorem of $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$
- sketch the graph of the periodic function.
- find the Laplace transform of the given function by using the theorem of $$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$
- find the inverse Laplace transform of the function by using the table of Laplace Transform
- find the inverse Laplace transform of the function by using the rules of Partial Fractions Decomposition
- solve the given initial value problem of the differential equation by using the method of Laplace Transforms
- find the transfer function and the impulse response function

Question 1

Find the Laplace transform of the given function by using the definition of Laplace Transform.

(a) $f(t) = 2t$

Solution:

$f(t) = 2t$

$$L\{2t\} = \int_0^\infty e^{-st}(2t)\ dt$$

$$= 2\int_0^\infty te^{-st}dt$$

$$= 2\left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= 2\left[(0-0)-(0-\frac{1}{s^2})\right]$$

$$= \frac{2}{s^2}$$

(b) $f(t) = e^{3t}$

Solution:

$f(t) = e^{3t}$

$$L\{e^{3t}\} = \int_0^\infty e^{-st}(e^{3t})\ dt$$

$$= \int_0^\infty e^{-t(s-3)}dt$$

$$= \left[\frac{e^{-t(s-3)}}{-(s-3)}\right]_0^\infty$$

$$= -\frac{1}{s-3}\left[e^{-t(s-3)}\right]_0^\infty$$

$$= -\frac{1}{s-3}[0-1]$$

$$= \frac{1}{s-3}$$

(c) $f(t) = \sin 2t$

<u>Solution:</u>

$f(t) = \sin 2t$

$L\{\sin 2t\} = \int_0^\infty e^{-5t} \sin 2t \ dt$

$\int u dv = uv - \int v du$

$dv = \sin 2t \ dt$

$v = \frac{-\cos 2t}{2}$

$u = e^{-5t}$

$\frac{du}{dt} = -se^{-5t}$

$du = -se^{-5t} dt$

$$\int e^{-5t} 2t \ dt = e^{-5t}(-\frac{\cos 2t}{2}) - \int (-\frac{\cos 2t}{2})(-se^{-5t} dt)$$

$$= \frac{-e^{-5t} \cos 2t}{2} - \frac{s}{2} \int e^{-5t} \cos 2t \ dt$$

$dv = \cos 2t \ dt$

$v = \frac{-\sin 2t}{2}$

$u = e^{-5t}$

$\frac{du}{dt} = -se^{-5t}$

$du = -se^{-5t} dt$

$$\int e^{-st} \sin 2t \ dt = \frac{e^{-st} \cos 3t}{2} - \frac{5}{2}\left[e^{-st}\left(\frac{\sin 2t}{2}\right) - \int \frac{\sin 2t}{2}\left(-se^{-st} dt\right)\right]$$

$$\int e^{-st} \sin 2t \ dt = \frac{e^{-st} \cos 2t}{2} - \frac{se^{-st} \sin 2t}{4} - \frac{s^2}{4} \int e^{-st} \sin 2t dt$$

$$\int e^{-st} \sin 2t \ dt + \frac{s^2}{4} \int e^{-st} \sin 2t \ dt = \frac{e^{-st} \cos 2t}{2} - \frac{se^{-st} \sin 2t}{4}$$

$$\left(1 + \frac{s^2}{4}\right) \int e^{-st} \sin 2t \ dt = \frac{e^{-st} \cos 2t}{2} - \frac{se^{-st} \sin 2t}{4}$$

$$\left(\frac{4+s^2}{4}\right) \int e^{-st} \sin 2t \ dt = \frac{-2e^{-st} \cos 2t - se^{-st} \sin 2t}{4}$$

$$\int e^{-st} \sin 2t \ dt = \frac{-2e^{-st} \cos 2t - se^{-st} \sin 2t}{4+s^2}$$

$$\int_0^\infty e^{-st}\sin 2t\ dt = \left[\frac{-2e^{-st}\cos 2t - se^{-st}\sin 2t}{4+s^2}\right]_0^\infty$$

$$= (0) - \left(\frac{-2(1)(1) - s(1)(0)}{4+s^2}\right)$$

$$= \frac{2}{4+s^2}$$

(d)

$$f(t) = \begin{cases} 0 & , \ 0 \le t \prec \pi \\ t-\pi & , \ \pi \le t \prec 2\pi \\ 0 & , \ t \succ 2\pi \end{cases}$$

<u>Solution:</u>

$$f(t) = \begin{cases} 0 & , \ 0 \le t \prec \pi \\ t-\pi & , \ \pi \le t \prec 2\pi \\ 0 & , \ t \succ 2\pi \end{cases}$$

$$L\{f(t)\} = \int_0^\pi e^{-st}0dt + \int_\pi^{2\pi} e^{-st}(t-\pi)dt + \int_{2\pi}^\infty e^{-st}0dt$$

$$= \int_\pi^{2\pi} te^{-st}dt - \int_\pi^{2\pi} \pi e^{-st}dt$$

$$= \int_\pi^{2\pi} te^{-st}dt + \left[\frac{\pi e^{-st}}{s}\right]_\pi^{2\pi}$$

$$\int_\pi^{2\pi} te^{-st}dt = \int udv$$

$$u = t \qquad dv = e^{-st}dt$$

$$\frac{du}{dt} = 1 \qquad v = \frac{e^{-st}}{-s}$$

$$du = dt$$

$$\int udv = uv - \int vdu$$

$$= \frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s}$$

$$= -\frac{te^{-st}}{s} - \int \frac{e^{-st}}{s}dt$$

$$= -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}$$

$$L\{f(t)\} = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2} + \frac{\pi e^{-st}}{s} \right]_0^{2\pi}$$

$$= \left(\frac{-2\pi e^{-2\pi s}}{s} - \frac{-e^{-2\pi s}}{s^2} + \frac{\pi e^{-2\pi s}}{s} \right) - \left(\frac{-\pi e^{-\pi s}}{s} - \frac{-e^{-\pi s}}{s^2} + \frac{\pi e^{-\pi s}}{s} \right)$$

$$= -\frac{-\pi e^{-2\pi s}}{s} - \frac{-e^{-2\pi s}}{s^2} + \frac{e^{-\pi s}}{s^2}$$

$$= \frac{-e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2}(\pi s + 1)$$

Question 2

Find the Laplace transform of each of the following functions.

(a) $y(t) = 2e^{4t} \sin 4t$

Solution:

$y(t) = 2e^{4t} \sin 4t = 2(e^{4t} \sin 4t)$

$L\{e^{at} f(t)\} = F(s-a)$

$f(t) = \sin 4t$

$$F(s) = \frac{4}{s^2 + 4^2}$$

$$F(s-4) = \frac{4}{(s-4)^2 + 4^2}$$

$$L\{y(t)\} = 2F(s-4) = 2\left(\frac{4}{(s-4)^2 + 4^2} \right) = \frac{8}{s^2 - 8s + 32}$$

(b) $y(t) = t^3 e^{-4t}$

Solution:

$y(t) = t^3 e^{-4t}$

$L\{e^{at} f(t)\} = F(s-a)$

$f(t) = t^3$

$$F(s) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$F(s-(-4)) = F(s+4) = \frac{6}{(s+4)^4}$$

$$L\{y(t)\} = F(s+4) = \frac{6}{(s+4)^4}$$

(c) $y(t) = t^2 - 3t - 2e^{-t}\cos 3t$

<u>Solution:</u>

$$y(t) = t^2 - 3t - 2e^{-t}\cos 3t$$

$$L\{y(t)\} = \frac{2!}{s^{2+t}} - 3\left(\frac{1!}{s^{1+t}}\right) - 2L\left\{e^{-t}\cos 3t\right\}$$

$$= \frac{2}{s^3} - \frac{3}{s^2} - 2L\left\{e^{-t}\cos 3t\right\}$$

$$y_1(t) = e^{-t}\cos 3t$$

$$L\{e^{at} f(t)\} = F(s-a)$$

$$f(t) = \cos 3t$$

$$F(s) = \frac{s}{s^2 + 3^2}$$

$$F(s+1) = \frac{s+1}{(s+1)^2 + 3^2} = \frac{s+1}{s^2 + 2s + 10}$$

$$L\{y_1(t)\} = F(s+1) = \frac{s+1}{(s+1)^2 + 3^2} = \frac{s+1}{s^2 + 2s + 10}$$

$$\therefore L\{y(t)\} = \frac{2}{s^3} - \frac{3}{s^2} - 2L\{y_1(t)\}$$

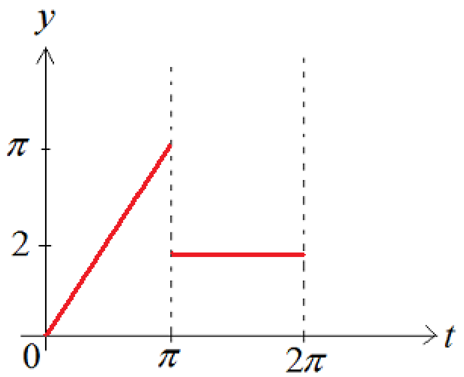
$$= \frac{2}{s^3} - \frac{3}{s^2} - \frac{2(s+1)}{s^2 + 2s + 10}$$

Question 3

Sketch the graph of the given step function, and write the function in terms of unit step function.

(a) $$f(t)=\begin{cases} t \ , & 0 \le t \prec \pi \\ 2 \ , & \pi \le t \prec 2\pi \\ 0 \ , & t \ge 2\pi \end{cases}$$

Solution:



$f_1(t) = t$

$t = 0 \ , \ y = 0 \qquad (0,0)$

$t = \pi \ , \ y = \pi \qquad (\pi, \pi)$

$f_2(t) = 2$

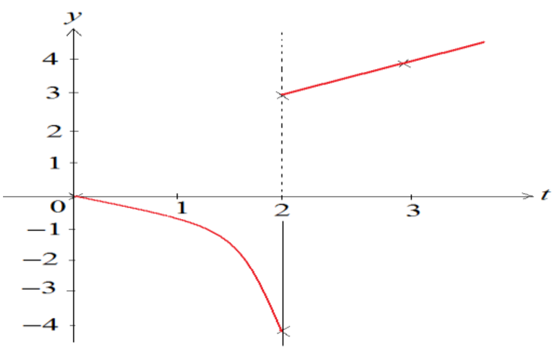
$$\boldsymbol{f(t) = f_1 + [f_2 - f_1]H(t - a_1) + [f_3 - f_2]H(t - a_2)}$$

$$f(t) = t + [2 - t]H(t - \pi) + [0 - 2]H(t - 2\pi)$$

$$= t + (2 - t)H(t - \pi) - 2H(t - 2\pi)$$

(b) $$f(t)=\begin{cases} -t^2 \ , & 0 \le t \prec 2 \\ 1 + t \ , & t \ge 2 \end{cases}$$

Solution:



$f_1(t) = -t^2$
$t = 0\ ,\ y = 0 \quad (0,0)$
$t = 2\ ,\ y = -4 \quad (2,-4)$

$f_2(t) = 1 + t$
$t = 2\ ,\ y = 3 \quad (2,3)$
$t = 3\ ,\ y = 4 \quad (3,4)$

$$\mathbf{f(t) = f_1 + [f_2 - f_1]H(t - a_1)}$$
$$f(t) = -t^2 + \left[1 + t - (-t^2)\right]H(t-2)$$
$$= -t^2 + \left[1 + t + t^2\right]H(t-2)$$

Question 4

Find the Laplace transform of the given function by using the theorem of $L\{f(t-a)H(t-a)\} = e^{-as}F(s)$

(a) $$f(t) = \begin{cases} 0 & ,0 \le t \prec 2 \\ (t-2)^2, & t \ge 2 \end{cases}$$

Solution:

STEP 1: Write the step function in terms of unit step function.

$$\mathbf{f(t) = f_1 + [f_2 - f_1]H(t - a_1)}$$
$$f(t) = 0 + \left[(t-2)^2 - 0\right]H(t-2)$$
$$= (t-2)^2 H(t-2)$$

STEP 2: Find $L\left\{(t-2)^2 H(t-2)\right\}$

$$f(t-2) = (t-2)^2$$
$$f(t) = t^2$$
$$F(s) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$$

$$L\{f(t-a)H(t-a)\} = e^{-as}F(s)$$
$$L\left\{(t-2)^2 H(t-2)\right\} = e^{-2s}\left(\frac{2}{s^3}\right) = \frac{2e^{-2s}}{s^3}$$

(b)

$$f(t)=\begin{cases} 0 \quad , \ 0 \le t \prec \pi \\ t-\pi \, , \pi \le t \prec 2\pi \\ 0 \quad , \quad t \ge 2\pi \end{cases}$$

<u>Solution:</u>

STEP 1: Write the unit step function in terms of unit step function.

$$\boldsymbol{f(t) = f_1 + [f_2 - f_1]H(t-a_1) + [f_3 - f_2]H(t-a_2)}$$

$$f(t) = 0 + \left[t - \pi - 0\right]H(t-\pi) + \left[0 - (t-\pi)\right]H(t-2\pi)$$

$$= (t-\pi)H(t-\pi) - (t-\pi)H(t-2\pi)$$

STEP 2: Find $L\left\{(t-\pi)H(t-\pi) - (t-\pi)H(t-2\pi)\right\}$

$$= L\left\{(t-\pi)H(t-\pi)\right\} - L\{(t-\pi)H(t-2\pi)\}$$

$$f(t-\pi) = t-\pi$$

$$f(t) = t$$

$$F(s) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

$$L\left\{f(t-a)H(t-a)\right\} = e^{-as}F(s)$$

$$L\left\{(t-\pi)H(t-\pi)\right\} = e^{-\pi s}\left(\frac{1}{s^2}\right) = \frac{e^{-\pi s}}{s^2}$$

$$f(t-2\pi) = t-\pi$$

$$= \left[(t-2\pi) + 2\pi\right] - \pi$$

$$f(t) = \left[t + 2\pi\right] - \pi$$

$$= t + \pi$$

$$F(s) = \frac{1!}{s^{1+1}} + \frac{\pi}{s} = \frac{1}{s^2} + \frac{\pi}{s}$$

$$L\left\{f(t-a)H(t-a)\right\} = e^{-as}F(s)$$

$$L\left\{(t-\pi)H(t-2\pi)\right\} = e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)$$

$$\therefore L\left\{(t-\pi)H(t-\pi) - (t-\pi)H(t-2\pi)\right\} = \frac{e^{-\pi s}}{s^2} - e^{-2\pi s}\left(\frac{1}{s^2} + \frac{\pi}{s}\right)$$

(c) $$f(t)=\begin{cases} 2\ ,\ 0\le t\prec 5 \\ 0\ ,5\le t\prec 10 \\ e^{4t}\ ,\quad t\ge 10\end{cases}$$

<u>Solution:</u>

STEP 1: Write the unit step function in terms of unit step function.

$$f(t)=2+(0-2)H(t-5)+(e^{4t}-0)H(t-10)$$

$$=2-2H(t-5)+(e^{4t}-0)H(t-10)$$

STEP 2: $L\left\{2-2H(t-5)+(e^{4t}-0)H(t-10)\right\}$

(i) $L\left\{2\right\}=\dfrac{2}{5}$

(ii) $L\left\{2H(t-5)\right\}$

$$f(t-5)=2$$

$$f(t)=2$$

$$F(s)=\frac{2}{s}$$

$$L\left\{2H(t-5)\right\}=e^{5s}\left(\frac{2}{s}\right)=\frac{2e^{5s}}{s}$$

(iii) $L\left\{2-2H(t-5)+(e^{4t})H(t-10)\right\}$

$$L\left\{(e^{4t})H(t-10)\right\}$$

$$f(t-10)=e^{4t}=e^{4[(t-10)+10]}$$

$$f(t)=e^{4t}=e^{4[t+10]}=e^{4t+40}$$

$$F(s)=e^{40}L\left\{e^{4t}\right\}=e^{40}\left(\frac{1}{s-4}\right)=\frac{e^{40}}{s-4}$$

$$L\left\{(e^{4t})H(t-10)\right\}=e^{-10s}\left(\frac{e^{40}}{s-4}\right)=e^{-10s+40}$$

$$\therefore L\left\{2-2H(t-5)+(e^{4t})H(t-10)\right\}=\frac{2}{s}-\frac{2e^{-5s}}{s}+e^{-10s+40}$$

(d)
$$f(t)=\begin{cases}\sin t, & 0\le t\prec \dfrac{\pi}{4}\\ \sin t+\cos(t-\dfrac{\pi}{4}), & t\ge \dfrac{\pi}{4}\end{cases}$$

<u>Solution:</u>

STEP 1: Write the unit step function in terms of unit step function.

$$f(t)=\sin t+\left[\sin t+\cos\left(t-\frac{\pi}{4}\right)-\sin t\right]H\left(t-\frac{\pi}{4}\right)$$
$$=\sin t+\cos\left(t-\frac{\pi}{4}\right)H\left(t-\frac{\pi}{4}\right)$$

STEP 2: $L\left\{\sin t+\cos\left(t-\frac{\pi}{4}\right)H\left(t-\frac{\pi}{4}\right)\right\}$

(i) $L\{\sin t\}=\dfrac{1}{s^2+1}$

(ii) $L\left\{\cos\left(t-\frac{\pi}{4}\right)H\left(t-\frac{\pi}{4}\right)\right\}$

$$f\left(t-\frac{\pi}{4}\right)=\cos\left(t-\frac{\pi}{4}\right)$$
$$f(t)=\cos t$$
$$F(s)=\frac{s}{s^2+1}$$
$$L\left\{\cos\left(t-\frac{\pi}{4}\right)H\left(t-\frac{\pi}{4}\right)\right\}=e^{\frac{-\pi s}{4}}\left(\frac{s}{s^2+1}\right)=\frac{se^{-\frac{\pi s}{4}}}{s^2+1}$$

$$L\left\{\sin t+\left[\sin t+\cos\left(t-\frac{\pi}{4}\right)-\sin t\right]H\left(t-\frac{\pi}{4}\right)\right\}$$
$$=L\left\{\sin t+\cos\left(t-\frac{\pi}{4}\right)H\left(t-\frac{\pi}{4}\right)\right\}$$
$$=\frac{1}{s^2+1}+\frac{se^{-\frac{\pi s}{4}}}{s^2+1}$$
$$=\frac{1+se^{-\frac{\pi s}{4}}}{s^2+1}$$

Question 5

Sketch the graph of the following periodic function, and find the $L\{f(t)\}$.

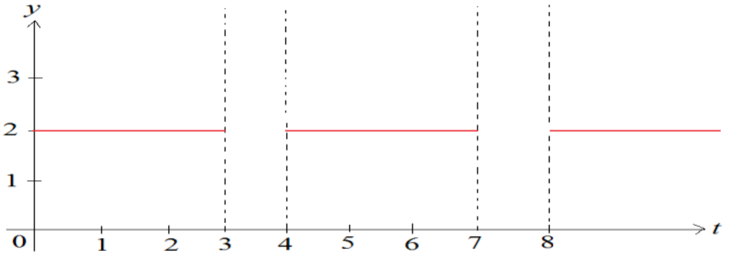
(a) Periodic pulse function:

$$f(t)=\begin{cases}2, 0\le t\prec 3\\ 0, 3\le t\ge 4\end{cases}$$

$$f(t)=f(t+4)$$

Solution:

$$f(t)=\begin{cases}2 \quad , \quad 0\le t\prec 3\\ 0 \quad , \quad 3\le t\prec 4\end{cases}$$



$$L\{f(t)\ \}=\frac{1}{1-e^{-4s}}\int_0^4 e^{-st}f(t)dt$$

$$\int_0^4 e^{-st}f(t)dt=\int_0^3 e^{-st}(2)dt+\int_3^4 e^{-st}(0)dt$$

$$=\left[\frac{2e^{-st}}{-s}\right]_0^3$$

$$=\frac{2e^{-3s}}{-s}+\frac{2}{s}$$

$$=\frac{2}{s}-\frac{2e^{-3s}}{s}$$

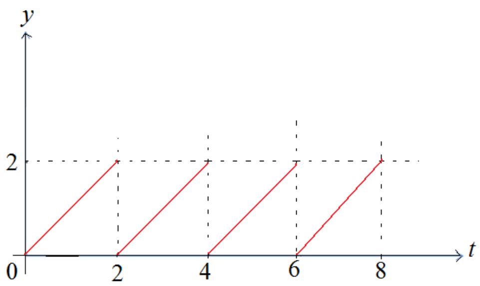
$$L\{f(t)\}=\frac{1}{1-e^{-4s}}\left(\frac{2}{s}-\frac{2e^{-3s}}{s}\right)=\frac{2(1-e^{-3s})}{s\left(1-e^{-4s}\right)}$$

(b) Saw-tooth function:

$f(t) = t, 0 \le t \prec 2$

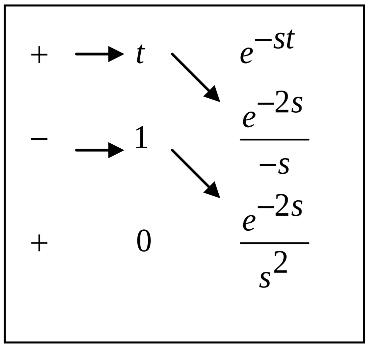
$$f(t) = f(t+2)$$

<u>Solution:</u>



$$\int_0^2 te^{-2s} = \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^2$$

$$= \frac{2e^{-st}}{-s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}$$



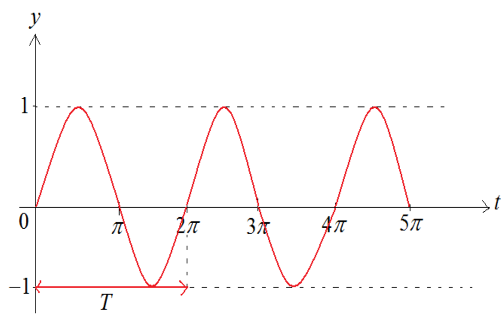
$$\therefore L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 te^{-st} dt = \frac{-2se^{-2s} - e^{-2s} + 1}{s^2\left(1-e^{-2s}\right)}$$

Question 6

Given $f(t) = \sin t, t \succ 0$

(a) Sketch the graph of $f(t)$

<u>Solution:</u>



(b) From the graph, find the period of function $f(t)$

<u>Solution:</u>

$T = 2\pi$

(c) By using the theorem $L\{f(t)\}=\dfrac{1}{1-e^{-sT}}\displaystyle\int_0^T e^{-st}f(t)dt$, show that $L\{\cos t\}=\dfrac{s}{s^2+1}$

Solution:

$$L\{\sin t\}=\frac{1}{1-e^{-2\pi s}}\int_0^{2\pi}e^{-st}\sin t\,dt$$

$+$	$\longrightarrow$	e^{-st}	$\sin t$
$-$	$\longrightarrow$	$-se^{-st}$	$-\cos t$
$+$	$\longrightarrow$	s^2e^{-st}	$-\sin t$
			$\int -\sin t$

$$\int_0^{2\pi} e^{-st}\sin t\,dt$$

$$=\left[-e^{-st}\cos t-se^{-st}\sin t\right]_0^{2\pi}-s^2\int_0^{2\pi} e^{-st}\sin t\,dt$$

$$\left(s^2+1\right)\int_0^{2\pi} e^{-st}\sin t\,dt=-e^{-2\pi s}+1$$

$$\int_0^{2\pi} e^{-st}\sin t\,dt=\frac{1-e^{-2\pi s}}{s^2+1}$$

$$\therefore L\{\sin t\}=\frac{1}{1-e^{-2\pi s}}\left(\frac{1-e^{-2\pi s}}{s^2+1}\right)$$

$$=\frac{1}{s^2+1}$$

Question 7

Find the inverse Laplace transform of the function.

(a) $\dfrac{3}{s^2+4}$

Solution:

$$L^{-1}\left\{\frac{3}{s^2+4}\right\}=\frac{3}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$=\frac{3}{2}\sin 2t$$

(b) $\dfrac{4}{(s-1)^3}$

Solution:

$$L^{-1}\left\{\frac{4}{(s-1)^3}\right\}$$

$$F(s-1)=\frac{4}{(s-1)^3}$$

$$F(s)=\frac{4}{s^3}$$

$$f(t)=L^{-1}\left\{\frac{4}{s^3}\right\}=2L^{-1}\left\{\frac{2!}{s^{2+1}}\right\}=2t^2$$

$$L^{-1}\{F(s-a)\}=e^{at}f(t)$$

$$L^{-1}\left\{\frac{4}{(s-1)^3}\right\}=L^{-1}\{F(s-a)\}=e^t\,2t^2=2e^t t^2$$

(c) $\dfrac{2}{s^2+3s-4}$

Solution:

$$L^{-1}\left\{\frac{2}{s^2+3s-4}\right\}$$

$$\frac{2}{s^2+3s-4}=\frac{2}{(s+4)(s-1)}$$

$$\frac{2}{(s+4)(s-1)}=\frac{A}{s+4}+\frac{B}{s-1}$$

$$2=A(s-1)+B(s+4)$$

$s=1$; $\quad 2=B(1+4)$

$$B=\frac{2}{5}$$

$s=-4$; $\quad 2=A(-4+1)$

$$A=-\frac{2}{5}$$

$$L^{-1}\left\{\frac{2}{s^2+3s-4}\right\}=L^{-1}\left\{\frac{-\frac{2}{5}}{S+4}\right\}+L^{-1}\left\{\frac{\frac{2}{5}}{S-1}\right\}$$

$$=-\frac{2}{5}L^{-1}\left\{\frac{1}{S+4}\right\}+\frac{2}{5}L^{-1}\left\{\frac{1}{S-1}\right\}$$

$$=-\frac{2}{5}e^{-4t}+\frac{2}{5}e^{t}$$

$$=\frac{2}{5}\left(e^{t}-e^{-4t}\right)$$

(d) $\frac{3s}{s^2-s-6}$

<u>Solution:</u>

$$\frac{3s}{s^2-s-6}=\frac{3s}{(s-3)(s+2)}=\frac{A}{(s-3)}+\frac{B}{(s+2)}$$

$$3s=A(s+2)+B(s-3)$$

$s=-2$, $3(-2)=0+B(-2-3)$

$$B=\frac{6}{5}$$

$s=3$, $3(3)=A(3+2)$

$$A=\frac{9}{5}$$

$$L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}=L^{-1}\left\{\frac{\frac{9}{5}}{S-3}\right\}+L^{-1}\left\{\frac{\frac{6}{5}}{s+2}\right\}$$

$$=\frac{9}{5}e^{3t}+\frac{6}{5}e^{-2t}$$

$$=\frac{3}{5}\left(3e^{3t}+2e^{-2t}\right)$$

(e) $$\frac{2s+2}{s^2+2s+5}$$

<u>Solution:</u>

$$\frac{2s+2}{s^2+2s+5}=\frac{2(s+1)}{(s+1)^2+4}=\frac{2(s+1)}{(s+1)^2+2^2}$$

$$F(s+1)=\frac{2(s+1)}{(s+1)^2+2^2}$$

$$F(s)=\frac{2s}{s^2+2^2}$$

$$f(t)=2\cos 2t$$

$$L^{-1}\left\{F(s-a)\right\}=e^{at}f(t)$$

$$L^{-1}\left(\frac{2s+2}{s^2+2s+5}\right)=e^{-t}(2\cos 2t)=2e^{-t}\cos 2t$$

(f) $$\frac{e^{-2s}}{s^2+s-2}$$

<u>Solution:</u>

$$\frac{e^{-2s}}{s^2+s-2}=\frac{e^{-2s}.1}{(s+2)(s-1)}=\left(\frac{A}{s-2}+\frac{B}{s-1}\right)e^{-2s}$$

$$1=A(s-1)+B(s+2)$$

$$s=1,\quad 1=B(1+2)$$

$$B=\frac{1}{3}$$

$$s=-2,\quad 1=A(-2-1)$$

$$A=-\frac{1}{3}$$

$$L^{-1}\left\{\frac{e^{-2s}}{s^2+s-2}\right\}=L^{-1}\left\{e^{-2s}\left(\frac{-\frac{1}{3}}{s+2}\right)\right\}+L^{-1}\left\{e^{-2s}\left(\frac{\frac{1}{3}}{s-1}\right)\right\}$$

$$=-\frac{1}{3}e^{-2(t-2)}H(t-2)+\frac{1}{3}e^{t-2}H(t-2)$$

$$=\frac{1}{3}H(t-2)\left[e^{-2(t-2)}+e^{t-2}\right]$$

(g) $\dfrac{2e^{-2s}}{s^2-4}$

<u>Solution:</u>

$$\frac{2e^{-2s}}{s^2-4}=\frac{2e^{-2s}}{s^2-2^2}=e^{-2s}.\frac{2}{s^2-2^2}$$

$$F(s)=\frac{2}{s^2-2^2}$$
$$f(t)=\sinh 2t$$

$$L^{-1}\left\{e^{-as}F(s)\right\}=f(t-a)H(t-a)$$
$$L^{-1}\left\{\frac{2e^{-2s}}{s^2-4}\right\}=\sinh 2(t-2).H(t-2)$$

(h) $\dfrac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}$

<u>Solution:</u>

$$L^{-1}\left\{\frac{e^{-as}}{s}\right\}=H(t-a)$$
$$L^{-1}\left\{\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right\}=L^{-1}\left\{\frac{e^{-s}}{s}+\frac{e^{-2s}}{s}-\frac{e^{-3s}}{s}-\frac{e^{-4s}}{s}\right\}$$
$$=H(t-1)+H(t-2)-H(t-3)-H(t-4)$$

(i) $\dfrac{2s-3}{s^2-4}$

<u>Solution:</u>

$$L^{-1}\left\{\frac{2s-3}{s^2-4}\right\}=2L^{-1}\left\{\frac{s}{s^2-2^2}\right\}-\frac{3}{2}L^{-1}\left\{\frac{2}{s^2-2^2}\right\}$$
$$=2\cosh 2t-\frac{3}{2}\sinh 2t$$

(j) $\dfrac{2s+1}{s^2-2s+2}$

<u>Solution:</u>

$$\frac{2s+1}{s^2-2s+2}=\frac{2(s-1)+3}{(s-1)^2+1}$$

$$F(s-1)=\frac{2(s-1)+3}{(s-1)^2+1}$$

$$F(s)=\frac{2s+3}{s^2+1^2}$$

$$f(t)=2L^{-1}\left\{\frac{s}{s^2+1^2}\right\}+3L^{-1}\left\{\frac{1}{s^2+1^2}\right\}$$

$$=2L^{-1}\left\{\frac{s}{s^2-2^2}\right\}-\frac{3}{2}L^{-1}\left\{\frac{s}{s^2-2^2}\right\}$$

$$=2\cos t+3\sin t$$

$$L^{-1}\{F(s-a)\}=e^{at}f(t)$$

$$L^{-1}\left\{\frac{2s+1}{s^2-2s+2}\right\}=e^{t}(2\cos t+3\sin t)$$

(k) $\dfrac{8s^2-4s+12}{s(s^2+4)}$

<u>Solution:</u>

$$\frac{8s^2-4s+12}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$$

$$8s^2-4s+12=A(s^2+4)+(Bs+C)s$$

$s=0$, $12=4A$

$A=3$

$s=1$, $8-4+12=A(1+4)+(B+C)$

$16=3(5)+B+C$

$B+C=1$ ……………………..(1)

$s=-1$, $8+4+12=3(1+4)+(-B+C)(-1)$

$24=15+B-C$

$B-C=9$ ………………………(2)

(2)+(1): $2B=10$

$B=5$

$$5+C=1$$
$$C=-4$$

$$L^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\}=L^{-1}\left\{\frac{3}{s}\right\}+L^{-1}\left\{\frac{5s-4}{s^2+4}\right\}$$
$$=3+5L^{-1}\left\{\frac{s}{s^2+2^2}\right\}-2L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$
$$=3+5\cos 2t-2\sin 2t$$

(l) $\dfrac{1-2s}{s^2+4s+5}$

Solution:

$$\frac{1-2s}{s^2+4s+5}=\frac{-2(s+2)+5}{(s+2)^2+1}$$
$$F(s+2)=\frac{-2(s+2)+5}{(s+2)^2+1}$$
$$F(s)=\frac{-2s+5}{s^2+1}$$
$$f(t)=-2\cos t+5\sin t$$

$$L^{-1}\{F(s-a)\}=e^{at}f(t)$$
$$L^{-1}\left\{\frac{1-2s}{s^2+4s+5}\right\}=e^{-2t}(-2\cos t+5\sin t)$$

(m) $\dfrac{2s-3}{s^2+2s+10}$

Solution:

$$\frac{2s-3}{s^2+2s+10}=\frac{2(s+1)-5}{(s-1)^2+9}=\frac{2(s+1)-5}{(s+1)+3^2}$$
$$F(s+1)=\frac{2(s+1)-5}{(s+1)+3^2}$$
$$F(s)=\frac{2s-5}{s^2+3^2}$$
$$f(t)=2\cos 3t-\frac{5}{3}\sin 3t$$

$$L^{-1}\left\{F(s-a)\right\}=e^{at}f(t)$$

$$L^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\}=e^{-t}\left(2\cos 3t-\frac{5}{3}\sin 3t\right)$$

(n) $\dfrac{s}{(s+1)(s^2+4)}$

Solution:

$$\frac{s}{(s+1)(s^2+4)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$s=A(s^2+4)+(Bs+C)(s+1)$$

$s=-1$, $\quad -1=A(1+4)$

$$A=-\frac{1}{5}$$

$s=0$, $\quad 0=-\frac{1}{5}(0+4)+(0+C)(1)$

$$C=\frac{4}{5}$$

$s=1$, $\quad 1=-\frac{1}{5}(1+4)+(B+\frac{4}{5})(2)$

$$1=-1+2B+\frac{8}{5}$$

$$2B=\frac{2}{5}$$

$$B=\frac{1}{5}$$

$$L^{-1}\left\{\frac{s}{(s+1)(s^2+4)}\right\}=L^{-1}\left\{\frac{-\frac{1}{5}}{s+1}\right\}+L^{-1}\left\{\frac{\frac{1}{5}s+\frac{4}{5}}{s^2+4}\right\}$$

$$=-\frac{1}{5}e^{-t}+\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$$

$$=\frac{1}{5}\left(-e^{-t}+\cos 2t+2\sin 2t\right)$$

Question 8

Solve the given initial value problem for $y(t)$ using the method of Laplace Transforms.

(a) $y''-7y'+10=0$; $y(0)=0$; $y'(0)=-3$

Solution:

$$L\{y''-7y'+10y\}=L\{0\}$$

$$s^2Y(s)-sy(0)-y'(0)-7\left[sY(s)-y(0)\right]+10Y(s)=0$$

$$s^2Y(s)-s(0)-(-3)-7sY(s)++7(0)+10Y(s)=0$$

$$\left[s^2-7s+10\right]Y(s)=-3$$

$$\begin{aligned} Y(s)&=\frac{-3}{s^2-7s+10} \\ &=\frac{-3}{(s-2)(s-5)} \\ &=\frac{A}{s-2}+\frac{B}{s-5} \end{aligned}$$

$$-3=A(s-5)+B(s-2)$$

$$s=5,\quad -3=3B$$
$$B=-1$$

$$s=2,\quad -3=-3A$$
$$A=1$$

$$\therefore Y(s)=\frac{1}{s-2}-\frac{1}{s-5}$$
$$y(t)=e^{2t}-e^{5t}$$

(b) $y''+9y=10e^{2t}$; $y(0)=-1$; $y'(o)=5$

Solution:

$$L\{y''+9y\}=L\{10e^{2t}\}$$

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=\frac{10}{s-2}$$

$$s^2Y(s)-s(-1)-5+9Y(s)=\frac{10}{s-2}$$

$$\left[s^2+9\right]Y(s)=\frac{10}{s-2}-s+5$$

$$Y(s)=\frac{10-s^2+2s+5s-10}{(s-2)(s^2+9)}$$
$$=\frac{-s^2+7s}{(s-2)(s^2+9)}$$
$$=\frac{A}{s-2}+\frac{Bs+C}{s^2+9}$$
$$-s^2+7s=A(s^2+9)+(Bs+C)(s-2)$$

$s=2$, $-2^2+7(2)=A(2^2+9)$
$$A=\frac{10}{13}$$
$s=0$, $0=\frac{10}{13}(9)+C(-2)$
$$C=\frac{45}{13}$$
$s=1$, $-1+7=\frac{10}{13}(10)+\left(B+\frac{45}{13}\right)(-1)$
$$B=-\frac{23}{13}$$

$$Y(s)=\frac{\left(\frac{10}{13}\right)}{s-2}+\frac{\left(-\frac{23}{13}\right)s+\frac{45}{13}}{s^2+9}$$
$$=\left(\frac{10}{13}\right)\left(\frac{1}{s-2}\right)-\left(\frac{23}{13}\right)\left(\frac{s}{s^2+3^2}\right)+\left(\frac{15}{13}\right)\left(\frac{3}{s^2+3^2}\right)$$
$$y(t)=\frac{10}{13}e^{2t}-\frac{23}{13}\cos 3t+\frac{15}{13}\sin 3t$$

(c) $y''-4y'+4y=te^t$; $y(0)=0$; $y'(o)=0$

<u>Solution</u>

$$L\{y''-4y'+4y\}=L\{te^t\}$$
$$s^2Y(s)-sy(0)-y'(0)-4\left[sY(s)-y(0)\right]+4Y(s)=\frac{1}{(s-1)^2}$$
$$\left[s^2-4s+4\right]Y(s)=\frac{1}{(s-1)^2}$$
$$Y(s)=\frac{1}{(s-1)^2(s-2)^2}$$
$$=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s-2}+\frac{D}{(s-2)^2}$$

$$1 = A(s-1)(s-2)^2 + B(s-2)^2 + C(s-1)^2(s-2) + D(s-1)$$

$s = 1$, $B = 1$

$s = 2$, $D = 2$

$s = 0$, $1 = A(-1)(4) + 1(4) + C(1)(-2) + 1(1)$

$4A + 2C = 4$ ……………………………………………(1)

$s = -1$, $1 = A(-2)(9) + 1(9) + C(4)(-3) + 1(4)$

$18A + 12C = 12$

$3A + 2C = 2$ ………..……………………………………(2)

(1)-(2): A=2

Substitute A=2 into (1),

$4(2) + 2C = 4$

$C = -2$

$$Y(s) = \frac{2}{s-1} + \frac{1}{(s-1)^2} - \frac{2}{s-2} + \frac{1}{(s-2)^2}$$

$$y(t) = 2e^t + te^t - 2e^{2t} + te^{2t}$$

(d) $y'' - 2y' + 5y = 1$; $y(0) = 0$; $y'(0) = 5$

<u>Solution:</u>

$$L\{y'' - 2y' + 5y\} = L\{1\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{s}$$

$$[s^2 - 2s + 5]Y(s) = \frac{1}{s} + 5$$

$$Y(s) = \frac{1+5s}{s(s^2 - 2s + 5)}$$

$$= \frac{A}{s} + \frac{Bs + C}{s^2 - 2s + 5}$$

$$1 + 5s = A(s^2 - 2s + 5) + (Bs + C)s$$

$s = 0$, $1 + 5(0) = A(0^2 - 2(0) + 5)$

$$A = \frac{1}{5}$$

$s = 1$, $1 + 5(1) = A(1^2 - 2(1) + 5)$

$$B + C = \frac{26}{5} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$s = -1$, $1 + 5(-1) = \frac{1}{5}(1 + 2 + 5) + (-B + C)(-1)$

$$B - C = -\frac{28}{5} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(1)+(2): $2B = -\frac{2}{5}$

$$B = -\frac{1}{5}$$

Substitute $B = -\frac{1}{5}$ into (1),

$$-\frac{1}{5} + C = \frac{26}{5}$$

$$C = \frac{27}{5}$$

$$Y(s) = \frac{\left(\frac{1}{5}\right)}{s} + \frac{\left(-\frac{1}{5}\right)s + \frac{27}{5}}{s^2 - 2s + 5}$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{5}\right)\left[\frac{(s-1) - 26}{s^2 - 2s + 5}\right]$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{5}\right)\left[\frac{(s-1) - 26}{(s-1)^2 + 2^2}\right]$$

$$= \left(\frac{1}{5}\right)\left(\frac{1}{s}\right) - \left(\frac{1}{5}\right)\left[\frac{s-1}{(s-1)^2 + 2^2}\right] + \left(\frac{13}{5}\right)\left(\frac{2}{(s-1)^2 + 2^2}\right)$$

$$y(t) = \frac{1}{5} - \frac{1}{5}e^t \cos 2t + \frac{13}{5}e^t \sin 2t$$

(e) $y''+3y'-4y=H(t-1)$; $y(0)=0$; $y'(o)=1$ where H is a unit step function.

Solution:

$$L\{y''+3y'-4y\}=L\{H(t-1)\}$$

$$s^2Y(s)-sy(0)-y'(0)+3[sY(s)-y(0)]-4Y(s)=\frac{e^{-s}}{s}$$

$$\left[s^2+3s-4\right]Y(s)=\frac{e^{-s}}{s}+1$$

$$Y(s)=\frac{e^{-s}}{s(s^2+3s-4)}+\frac{1}{s^2+3s-4}$$

(i) $$L^{-1}\left\{\frac{e^{-s}}{s(s^2+3s-4)}\right\}=f(t-1)H(t-1)$$

$$a=1,\quad F(s)=\frac{1}{s(s^2+3s-4)}$$

$$=\frac{1}{s(s+4)(s-1)}$$

$$=\frac{A}{s}+\frac{B}{s+4}+\frac{C}{s-1}$$

$$1=A(s+4)(s-1)+B(s)(s-1)+Cs(s+4)$$

$$s=0,\quad 1=A(4)(-1)$$

$$A=-\frac{1}{4}$$

$$s=1,\quad 1=C(1)(5)$$

$$C=\frac{1}{5}$$

$$s=-4,\quad 1=B(-4)(-5)$$

$$B=\frac{1}{20}$$

$$F(s)=\frac{\left(-\frac{1}{4}\right)}{s}+\frac{\left(\frac{1}{20}\right)}{s+4}+\frac{\left(\frac{1}{5}\right)}{s-1}$$

$$f(t)=-\frac{1}{4}+\frac{1}{20}e^{-4t}+\frac{1}{5}e^{t}$$

$$L^{-1}\left\{\frac{e^{-s}}{s(s^2+3s-4)}\right\}=\left[-\frac{1}{4}+-\frac{1}{20}e^{-4(t-1)}+\frac{1}{5}e^{t-1}\right]H(t-1)$$

(ii) $L^{-1}\left\{\frac{1}{s^2+3s-4}\right\} = L^{-1}\left\{\frac{1}{(s+4)(s-1)}\right\}$

$$= L^{-1}\left\{\frac{A}{s+4}+\frac{B}{s-1}\right\}$$

$$= L^{-1}\left\{\frac{\left(-\frac{1}{5}\right)}{s+4}+\frac{\left(\frac{1}{5}\right)}{s-1}\right\}$$

$$= -\frac{1}{5}e^{-4t}+\frac{1}{5}e^{t}$$

$$\therefore y(t) = L^{-1}\left\{\frac{e^{-s}}{s(s^2+3s-4)}\right\}+L^{-1}\left\{\frac{1}{s^2+3s-4}\right\}$$

$$= \left[-\frac{1}{4}+-\frac{1}{20}e^{-4(t-1)}+\frac{1}{5}e^{t-1}\right]H(t-1)-\frac{1}{5}e^{-4t}+\frac{1}{5}e^{t}$$

Question 9

Find the transfer function and the impulse response function.

(a) $y''-7y'+10y = f(t)$

Solution:

$$s^2Y(s)-sy(0)-y'(0)-7\left[sY(s)-y(0)\right]+10Y(s) = F(s)$$

$$\left[s^2-7s+10\right]Y(s) = F(s)$$

$$\frac{Y(s)}{F(s)} = \frac{1}{s^2-7s+10}$$

$$G(s) = \frac{1}{(s-2)(s-5)}$$

$$g(t) = L^{-1}\left\{\frac{1}{(s-2)(s-5)}\right\}$$

$$= L^{-1}\left\{\frac{A}{s-2}+\frac{B}{s-5}\right\}$$

$$1 = A(s-5)+B(s-2)$$

$s=2,\ \ 1=-3A$

$$A=-\frac{1}{3}$$

$s=5,\ \ 1=3B$

$$B=\frac{1}{3}$$

$$g(t)=-\frac{1}{3}e^{2t}+\frac{1}{3}e^{5t}$$

(b) $y''+9y=f(t)$

Solution:

$$s^2Y(s)-sy(0)-y'(0)+9Y(s)=F(s)$$

$$\left[s^2+9\right]Y(s)=F(s)$$

$$\frac{Y(s)}{F(s)}=\frac{1}{s^2+9}$$

$$G(s)=\frac{1}{s^2+9}$$

$$g(t)=L^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$=\frac{1}{3}\sin 3t$$

(c) $y''-4y'+4y=f(t)$

Solution:

$$s^2Y(s)-sy(0)-y'(0)-4\left[sY(s)-y(0)\right]+4Y(s)=F(s)$$

$$\left[s^2-4s+4\right]Y(s)=F(s)$$

$$\frac{Y(s)}{F(s)}=\frac{1}{s^2-4s+4}$$

$$G(s)=\frac{1}{(s-2)^2}$$

$$g(t)=L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$=e^{2t}t=te^{2t}$$

(d) $y''-2y'+5y=f(t)$

<u>Solution:</u>

$$s^2Y(s)-sy(0)-y'(0)-2\left[sY(s)-y(0)\right]+5Y(s)=F(s)$$

$$\left[s^2-2s+5\right]Y(s)=F(s)$$

$$\frac{Y(s)}{F(s)}=\frac{1}{s^2-2s+5}$$

$$G(s)=\frac{1}{s^2-2s+5}$$

$$\begin{aligned}g(t)&=L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}\\&=L^{-1}\left\{\frac{1}{(s-1)^2+2^2}\right\}\\&=\frac{1}{2}L^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}\\&=\frac{1}{2}e^t\sin 2t\end{aligned}$$