CHAPTER TWO

MATRICES AND SYSTEMS OF LINEAR EQUATIONS

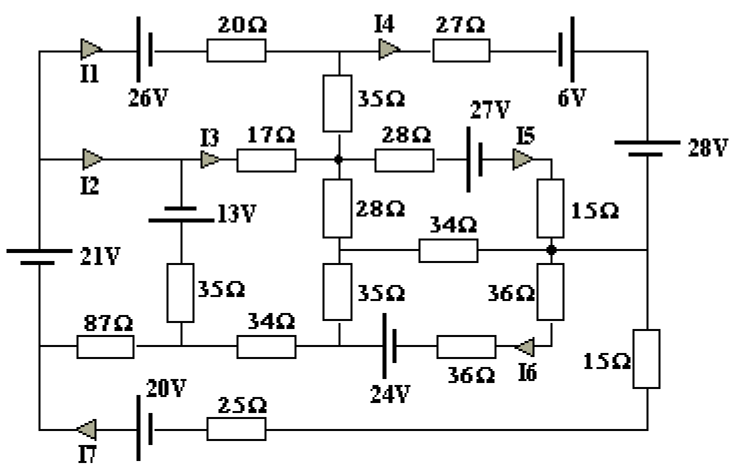
2.0 Introduction

A *matrix* is an ordered array of numbers. Matrices have many applications in science, engineering and computing.

These applications include:

- electronics (finding the currents in a circuit):

Using matrices, we can solve the currents I_1, I_2, I_3, I_4, I_5, I_6 and I_7 in the following circuit:



Circuit Equations:

$-26 = 72I_1 - 17I_3 - 35I_4$

$34 = 122I_2 - 35I_3 - 87I_7$

$-13 = 149I_3 - 17I_1 - 35I_2 - 28I_5 - 35I_6 - 34I_7$

$5 = 105I_4 - 35I_1 - 43I_5$

$-27 = 105I_5 - 28I_3 - 43I_4 - 34I_6$

$24 = 141I_6 - 35I_3 - 34I_5 - 72I_7$

$-4 = 233I_7 - 87I_2 - 34I_3 - 72I_6$

2.1 Matrix

A matrix is written with ( ) or [ ] brackets. Do not confuse a **matrix** with a **determinant** which uses | |. A **matrix** is a *pattern of numbers*; a **determinant** gives a *single number*.

The *size* of a matrix is written: rows × columns.

$\begin{bmatrix} 2 & 4 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ 2 × 3 matrix.

$\begin{bmatrix} 4 \\ 2.6 \\ -7 \end{bmatrix}$ 3 x 1 matrix

$\begin{bmatrix} 1 & 2 & 3 \\ 8 & 7 & 6 \\ -4 & 5 & 9 \end{bmatrix}$ 3 x 3 (square) matrix

Equal matrices have identical corresponding elements.

If $\begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & a \end{bmatrix}$

then $a = 3$, $x = 2$ and $y = 7$.

2.2 Identity Matrix

The Identity Matrix, called *I*, is a square matrix with all elements 0 except the *principal diagonal* which has all ones:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 2 x 2 identity matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3 x 3 identity matrix

2.3 Elements in a matrix

The **elements in a matrix *A*** are denoted by a_{ij}, where *i* is the row number and *j* is the column number.

In the matrix

$$A = \begin{bmatrix} 5 & 4 & 9 \\ -1 & 2 & 8 \end{bmatrix},$$

the element $a_{13} = 9$, since the element in the 1st row and 3rd column is 9.

2.4 Matrix Operations

2.4.1 Addition (and Subtraction) of Matrices

To add matrices, just add corresponding elements:

$$\begin{bmatrix} 8 & 3 & 4 \\ 0 & -1 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -2 & 1 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 8+5 & 3+-2 & 4+1 \\ 0+6 & -1+3 & 9+5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 1 & 5 \\ 6 & 2 & 14 \end{bmatrix}$$

Note: All matrices are 2×3.

2.4.2 Scalar Multiplication (and Division)

Scalar multiplication of matrices is similar to scalar multiplication of **vectors**. We multiply (or divide) each element by the scalar:

If

$$A = \begin{bmatrix} 3 & 1 \\ 7 & -1 \\ 2 & 8 \end{bmatrix}$$

then

$$5A = \begin{bmatrix} 5 \times 3 & 5 \times 1 \\ 5 \times 7 & 5 \times -1 \\ 5 \times 2 & 5 \times 8 \end{bmatrix} = \begin{bmatrix} 15 & 5 \\ 35 & -5 \\ 10 & 40 \end{bmatrix}$$

Again **note** that all are the same size (3 × 2).

2.4.3 Multiplication of Matrices

Important: We can only multiply matrices if the number of columns in the first matrix is the same as the number of rows in the second.

- Multiplying a 2 × 3 matrix by a 3 × 4 matrix is possible and it gives a 2 × 4 matrix as the answer.
- 7 × 1 times 1 × 2 okay; gives 7 × 2
- 4 × 3 times 2 × 3 is NOT possible.

The process: We multiply and add the elements as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

In general, when multiplying matrices, the commutative law doesn't hold, i.e. AB ≠ BA. There are two common exceptions to this:

- The identity matrix: $IA = AI = A$.
- The *inverse* of a matrix: $A^{-1}A = AA^{-1} = I$.

2.5 Determinants

A *determinant* is a square array of numbers (written within a pair of vertical lines) which represents a certain sum of products. It produces a single number (a *scalar* quantity).

2.5.1 2x2 Determinants

In general, we find the value of a 2×2 determinant as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

2.5.2 3×3 Determinants

A 3×3 determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

can be evaluated in various ways.

We will use...

Expansion by Minors

Using this method, we have:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - a_2 \begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

Note: The determinant $\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}$ is called the *minor* of a_1.

It is formed from the elements not in that row and not in that column.

The determinant $\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix}$ is called the *minor* of a_2.

It is formed from the elements not in that row and not in that column. Minors are used in certain matrix operations.

2.6 Finding the Inverse of a Matrix

2.6.1 Method 1 (only good for 2 × 2 matrices)

Find the inverse, A^{-1}, of

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix}$$

using Method 1.

Answer

[1] Interchange leading diagonal elements:

-7 → 2; 2 → -7

$$\begin{bmatrix} -7 & -3 \\ 4 & 2 \end{bmatrix}$$

[2] Change signs of the other 2 elements:

-3 → 3; 4 → -4

$$\begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix}$$

[3] Find |A|

$$\begin{bmatrix} 2 & -3 \\ 4 & -7 \end{bmatrix} = -14 + 12 = -2$$

[4] Multiply result of [2] by $\frac{1}{|A|}$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -7 & 3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 3.5 & -1.5 \\ 2 & -1 \end{bmatrix}$$

Is it correct?

Check:

$$A^{-1}A = \begin{bmatrix} 3.5 & -1.5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -7 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7-6 & -10.5+10.5 \\ 4-4 & -6+7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

2.6.2 Method 2 : Adjoint Matrix Method

If you need to find the inverse of a 3×3 (or bigger) matrix using paper, then follow the steps given. It is tedious, but it will get you there.

Method 2 uses the ***adjoint matrix*** method.

The inverse of a 3×3 matrix is given by:

$$A^{-1} = \frac{adjA}{detA}$$

We use **cofactors** to determine the **adjoint** of a matrix.

Cofactors

Recall: The *cofactor* of an element in a matrix is the value obtained by evaluating the determinant formed by the elements not in that particular row or column, and multiply with + or – sign.

Consider the matrix

$$\begin{bmatrix} \mathbf{5} & \mathbf{6} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-3} \\ \mathbf{4} & \mathbf{-7} & \mathbf{2} \end{bmatrix}$$

The minor of 6 is

$$\begin{bmatrix} 0 & -3 \\ 4 & 2 \end{bmatrix} = 0 - -12 = 12$$

The minor of -3 is

$$\begin{bmatrix} 5 & 6 \\ 4 & -7 \end{bmatrix} = -35 - 24 = -59$$

We find the **cofactor matrix** by replacing each element in the matrix with its minor and applying a + or - sign as follows:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Then we find the adjoint matrix by finding the **transpose** of the resulting matrix

[1st column becomes 1st row;

2nd column becomes 2nd row, etc].

2.6.3 Method 3: Gauss-Jordan Method

Basic Row Operations

1. Any two rows may be interchanged - order of equations doesn't matter.
2. Every element in any row may be multiplied by any given number other than zero multiply both sides of an equation by k.
3. Any row may be replaced by a row whose elements are the sum of a non-zero multiple of itself and a nonzero multiple of another row.

The basic plot to the Gauss-Jordan Method is to transform the augmented matrix [A|I] to [I|A^{-1}].

$$[\mathbf{A} \mid \mathbf{I}] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 5 & -1 & 0 & 1 & 0 \\ -2 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 \\ 0 & 3 & -4 & 2 & 0 & 1 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -5 & 2 & 0 \\ 0 & 1 & -2 & 3 & -1 & 0 \\ 0 & 0 & 2 & -7 & 3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{2} & \frac{-5}{2} & \frac{-3}{2} \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & \frac{-7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}]$$

2.7 Solving a System of Linear Equations

2.7.1 Method 1 : Inverse Matrix Method

2.7.1.1 Inverse Matrix Method for 2×2 Systems

The solution (x,y) of the system

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

2.7.1.2 Inverse Matrix Method for 3×3 Systems

We can solve the system

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

by using:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

2.7.2 Method 2: Cramer's Rule.

2.7.2.1 Cramer's Rule for 2×2 Systems

The solution (x,y) of the system

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

is given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

2.7.2.2 Cramer's Rule for 3×3 Systems

We can solve the system

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

by using:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\Delta}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\Delta}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\Delta}$$

Where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

2.7.3 Method 3 : Gaussian Elimination

Gaussian elimination gives us tools to solve large linear systems numerically. It is done by manipulating the given matrix using the elementary row operations to put the matrix into row echelon form. To be in row echelon form, a matrix must conform to the following criteria:

1. If a row does not consist entirely of zeros, then the first non zero number in the row is a 1.(the leading 1)
2. If there are any rows entirely made up of zeros, then they are grouped at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right that the leading 1 in the higher row.

From this form, the back substitution is easily applied

Since the matrix is representing the coefficients of the given variables in the system, the augmentation now represents the values of each of those variables. Consider the following example:

2.7.3.1 Consistent linear system, unique solution

Solve the system by using Gaussian Elimination method:

$$x + y + 2z = 8$$
$$-x - 2y + 3z = 1$$
$$3x - 7y + 4z = 10$$

Start With:	Elementary Row Operation(S)	Product
$\begin{array}{l} x + y + 2z = 8 \\ -1x - 2y + 3z = 1 \\ 3x - 7y + 4z = 10 \end{array}$	Place into augmented matrix	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$	R2 - (-1)R1 --> R2 R3 - (3)R1 --> R3	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$	(-1)R2 --> R2 R3 - (-10)R2 --> R3	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right] \quad \text{(-1/52)R3 --> R3} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

In Row Echelon Form --->

Now, let us apply back substitution:

$$z = 2$$

$$y - 5z = -9$$
$$y - 5(2) = -9$$
$$y = 1$$

$$x + y + 2z = 8$$
$$x + 1 + 2(2) = 8$$
$$x = 3$$

Therefore the solution is (3,1,2)

2.7.3.2 Inconsistent linear system

Solve the system by using Gaussian Elimination method:

$$x + y = 1$$
$$-x + y = 1$$
$$2x + 4y = 5$$
$$3x + 3y = 6$$

Start With:	**Elementary Row Operation(S)**	**Product**

$$\begin{aligned} x+y&=1 \\ -x+y&=1 \\ 2x+4y&=5 \\ 3x+3y&=6 \end{aligned}$$

Place into augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right]$$

```
R2 -(-1) R1 -->  R2
R3 - ( 2)R1 -->  R3
    R4-(3)R1 -->  R4
```

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{array}\right]$$

```
R3-R2 --> R3
```

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$$

$\frac{1}{2}$R2 --> R2

R4-(3)R3 --> R4

In Row Echelon Form --->

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Now, let us apply back substitution:

Row 3: 0x+0y=1 which is impossible. Therefore the system has no solution(inconsistent).

2.7.3.3 Consistent linear system, Infinitely Many Solutions

Solve the system by using Gaussian Elimination method:

$$\begin{aligned} x-2y+4z&=1 \\ -2x+5y+5z&=-1 \\ 5x-12y-6z&=3 \end{aligned}$$

Start With:	**Elementary Row Operation(S)**	**Product**

$$\begin{aligned} x-2y+4z&=1\\ -2x+5y+5z&=-1\\ 5x-12y-6z&=3 \end{aligned}$$

Place into augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1\\ -2 & 5 & 5 & -1\\ 5 & -12 & -6 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1\\ -2 & 5 & 5 & -1\\ 5 & -12 & -6 & 3 \end{array}\right]$$

```
R2 - (-2)R1 --> R2
R3 - ( 5)R1 --> R3
```

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1\\ 0 & 1 & 13 & 1\\ 0 & -2 & -26 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1\\ 0 & 1 & 13 & 1\\ 0 & -2 & -26 & -2 \end{array}\right]$$

```
R3 - (-2)R2 --> R3
In Row Echelon Form --->
```

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 1\\ 0 & 1 & 13 & 1\\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, let us apply back substitution:

Row 2:

$$\begin{aligned} &y+13z=1\\ &Let\\ &z=t\\ &then\\ &y=1-13t \end{aligned}$$

Row 1:

$$\begin{aligned} &x-2y+4z=1\\ &x=1+2(1-13t)-4t=3-30t \end{aligned}$$

Therefore the solution set is $(3-30t, 1-13t, t)$

2.7.4 Method 4 : Gauss – Jordan Elimination

This is a continuation of Gaussian elimination. To solve the system by this method we need to manipulate the augmented matrix by elementary row operations to put the matrix into reduced row echelon form.

For a matrix to be in reduced row echelon form, it must be in row echelon form and submit to one added criteria:

- Each column that contains a “leading 1” **has zeros above each “leading 1”.**

Consider the following example:

2.7.4.1 Consistent linear system, unique solution

Solve the system by using Gauss-Jordan Elimination method:

$$x + y + 2z = 8$$
$$-1x - 2y + 3z = 1$$
$$3x - 7y + 4z = 10$$

Start With:	**Elementary Row Operation(S)**	**Product**
$\begin{aligned} x + y + 2z &= 8 \\ -1x - 2y + 3z &= 1 \\ 3x - 7y + 4z &= 10 \end{aligned}$	Place into augmented matrix	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right]$	R2 - (-1)R1 --> R2 R3 - (3)R1 --> R3	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array}\right]$	(-1)R2 --> R2 R3 - (-10)R2 --> R3	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{array}\right]$	(-1/52)R3 --> R3 In Row Echelon Form --->	$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array}\right]$	R2 - (-5)R3 --> R2 R1 - (2)R3 --> R1	$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$
$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$	R1 - (1)R2 --> R1 Reduced Row Echelon Form --->	$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$

It is now obvious, by inspection, that the solution to this linear system is x=3, y=1, and z=2. Again, by solution, it is meant the x, y, and z required to satisfy all the equations simultaneously.

2.7.4.2 Inconsistent linear system

Solve the system by using Gauss-Jordan Elimination method:

$$\begin{aligned} x + y &= 1 \\ -x + y &= 1 \\ 2x + 4y &= 5 \\ 3x + 3y &= 6 \end{aligned}$$

Start With:	**Elementary Row Operation(S)**	**Product**
$\begin{aligned} x + y &= 1 \\ -x + y &= 1 \\ 2x + 4y &= 5 \\ 3x + 3y &= 6 \end{aligned}$	Place into augmented matrix	$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right]$
$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 3 & 6 \end{array}\right]$	R2 -(-1) R1 --> R2 R3 - (2)R1 --> R3 R4-(3)R1 --> R4	$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{array}\right]$
$\left[\begin{array}{cc\|c} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{array}\right]$	R3-R2 --> R3 R1-$\frac{1}{2}$R2 --> R1	$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$
$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{array}\right]$	$\frac{1}{2}$R2-R3 --> R2 R4-(3)R3 --> R4 In Reduced Row Echelon Form --->	$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$

Now, let us apply back substitution:

Row 3: 0x+0y=1 which is impossible. Therefore the system has no solution(inconsistent).

2.7.4.3 Consistent linear system, Infinitely Many Solutions

Solve the system by using Gauss-Jordan Elimination method:

$$\begin{aligned} x-2y+4z&=1\\ -2x+5y+5z&=-1\\ 5x-12y-6z&=3 \end{aligned}$$

Start With:	Elementary Row Operation(S)	Product
$\begin{aligned} x-2y+4z&=1\\ -2x+5y+5z&=-1\\ 5x-12y-6z&=3 \end{aligned}$	Place into augmented matrix	$\left[\begin{array}{ccc\|c} 1 & -2 & 4 & 1\\ -2 & 5 & 5 & -1\\ 5 & -12 & -6 & 3 \end{array}\right]$
$\left[\begin{array}{ccc\|c} 1 & -2 & 4 & 1\\ -2 & 5 & 5 & -1\\ 5 & -12 & -6 & 3 \end{array}\right]$	R2 - (-2)R1 --> R2 R3 - (5)R1 --> R3	$\left[\begin{array}{ccc\|c} 1 & -2 & 4 & 1\\ 0 & 1 & 13 & 1\\ 0 & -2 & -26 & -2 \end{array}\right]$
$\left[\begin{array}{ccc\|c} 1 & -2 & 4 & 1\\ 0 & 1 & 13 & 1\\ 0 & -2 & -26 & -2 \end{array}\right]$	R1-(-2)R2 --> R1 R3 - (-2)R2 --> R3 In Reduced Row Echelon Form --->	$\left[\begin{array}{ccc\|c} 1 & 0 & 30 & 3\\ 0 & 1 & 13 & 1\\ 0 & 0 & 0 & 0 \end{array}\right]$

Now, let us apply back substitution:

Row 2: $y+13z=1$, *Let* $z=t$, *then* $y=1-13t$

Row 1: $x+30z=3$, $x=3-30t$

Therefore the solution set is $(3-30t, 1-13t, t)$

There are some problems that could arise while searching for these solutions. If the lines are parallel then they will not intersect and thus provide no solution. In three dimensions the problem of skewing is possible. Lines are skewed if they lie in parallel planes yet have different slopes. If this problem occurs, it will made evident in the matrix by a row (or more than one) of zeros being present when the matrix is in row echelon form. Another problem that may arise is a division by zero. If a zero is placed in the main diagonal of the row being operated on, when you divide that row by the diagonal number the division by zero error will occur. To trap this error, simply check the diagonal number being worked with. If it is zero, exchange that row with the row below it. Exchanging rows is a legal elementary row operation.

2.8 Matrix Eigenvalue Problems

Definition:

*Let **A** be an* $n \times n$ *matrix. A real or complex number* λ *is called an eigenvalue of **A** if the matrix equation*

$$\mathbf{Ax} = \lambda \mathbf{x}$$

is satisfied for some nonzero vector x. The vector x is called an eigenvector of A associated with the eigenvalue λ *.*

Theorem:

The eigenvalues λ *of a square matrix A are the roots of the characteristic equation of A,*

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$