CHAPTER TWO

MULTIPLE INTEGRAL

After completing these tutorials, students should be able to:

- evaluate the double integral over the given rectangular region R
- find the volume of the solid bounded by the planes
- find the area of the region bounded by the curves using double integral
- find the volume of the solid bounded by the graphs using double integral
- find the area of the region by using double integral in polar coordinates
- change the integrand from Cartesian to polar coordinates
- calculate $\iiint_G f(x,y,z)\,dV$ over the given region G
- find the volume of the solid bounded by the planes using triple integral
- find the volume of the solid bounded by the given surfaces using triple integrals in cylindrical coordinate
- find the volume of the solid bounded by the planes given using triple integrals in spherical coordinate
- find the mass of the lamina region R
- find the mass and the center of mass of lamina region R bounded by the given graph and its density
- find the centroid for the given region

Question 1

Evaluate the double integral

$$\iint_R y\sqrt{1+y^2}\,dA$$

over the rectangular region $R = \{(x, y) : 0 \le x \le 1, 2 \le y \le 3\}$.

Solution:

$$\iint_R y\sqrt{1+y^2}\,dA = \int_0^1\int_2^3 y\sqrt{1+y^2}\,dy\,dx$$

Substitution Method:

$$u = 1+y^2 \qquad \frac{du}{dy} = 2y$$

$$dy = \frac{du}{2y}$$

$$= \int_0^1\left[\int_2^3 y(u)^{\frac{1}{2}}\frac{du}{2y}\right]dx$$

$$= \int_0^1\left[\frac{1}{2}\cdot(u)^{\frac{3}{2}}\cdot\frac{2}{3}\right]_2^3 dx$$

$$= \frac{1}{3}\int_0^1\left[\left(1+y^2\right)^{\frac{3}{2}}\right]_2^3 dx$$

$$= \frac{1}{3}\int_0^1\left[(10)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right]dx$$

$$= \frac{1}{3}\left[(10)^{\frac{3}{2}}x - (5)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{3}\left[(10)^{\frac{3}{2}} - (5)^{\frac{3}{2}}\right]$$

$$= 6.814$$

Question 2

Find the volume of the solid bounded by the plane $z = x^2$, $x = 2, y = 3$ and coordinate plane.

Solution

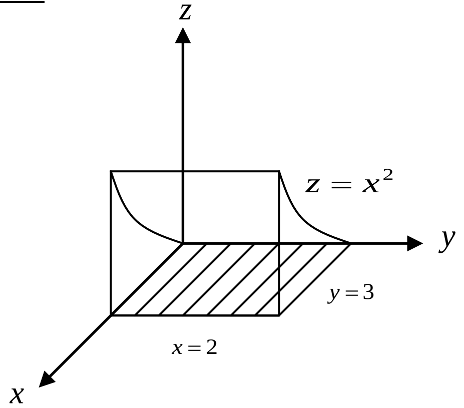


Figure 6.1 (a)

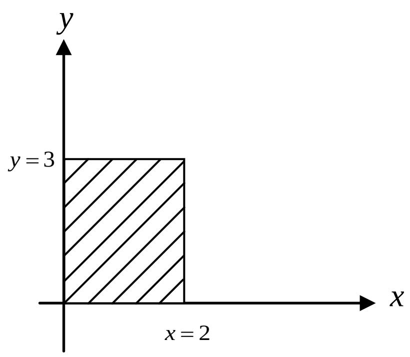


Figure 6.1 (b)

Volume of a solid is given by,

$$V = \iint_R f(x,y)dA \text{ where } f(x,y) = z:$$

Therefore,

$$V = \iint_R f(x,y)dA = \int_0^2 \int_0^3 z dy dx$$

$$= \int_0^2 \int_0^3 x^2 dy dx$$

$$= \int_0^2 \left[x^2 y \right]_0^3 dx$$

$$= \int_0^2 \left[3x^2 \right] dx$$

$$= \left[\frac{3x^2}{3} \right]_0^2$$

$$= 8 \text{ unit}^3.$$

Question 3

Evaluate $\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dydx$

Solution:

Substitution Method:

$$\int_0^1\int_0^x y\sqrt{x^2-y^2}\,dydx \longrightarrow u=y^2 \qquad \frac{du}{dy}=2y$$

$$=\int_0^1\int_0^x\left[y\sqrt{x^2-u}\right]\frac{du}{2y}dx \longleftarrow dy=\frac{du}{2y}$$

$$=\int_0^1\left[-\frac{1}{2}\left(x^2-u\right)^{\frac{3}{2}}\cdot\frac{2}{3}\right]_0^x dx$$

$$=\int_0^1\left[-\frac{1}{3}\left(x^2-y^2\right)^{\frac{3}{2}}\right]_0^x dx$$

$$=-\frac{1}{3}\int_0^1\left[\left(x^2-x^2\right)^{\frac{3}{2}}-\left(x^2-0\right)^{\frac{3}{2}}\right]dx$$

$$=-\frac{1}{3}\int_0^1\left[-x^3\right]dx$$

$$=\frac{1}{3}\left[\frac{x^4}{4}\right]_0^1$$

$$=\frac{1}{12}$$

Question 4

Let R be a region in the *xy*-plane and bounded by $y=\sqrt{x}, x=4, y=0$. Evaluate $\iint_R (xy)dA$.

Solution

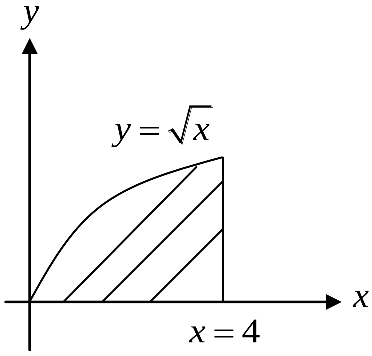


Figure 6.2

$$\iint_R (xy)\,dA = \int_0^4 \int_0^{\sqrt{x}} (xy)\,dydx$$
$$= \int_0^4 \left(\frac{xy^2}{2}\right)_0^{\sqrt{x}} dx$$
$$= \int_0^4 \left(\frac{x^2}{2}\right)_0^{\sqrt{x}} dx$$
$$= \left(\frac{x^3}{6}\right)_0^4$$
$$= \frac{32}{3}$$

Question 5

By using the double integral, find the area of the region bounded by the curves below:
$y=-x^2+9, x=0, y=0$

Solution

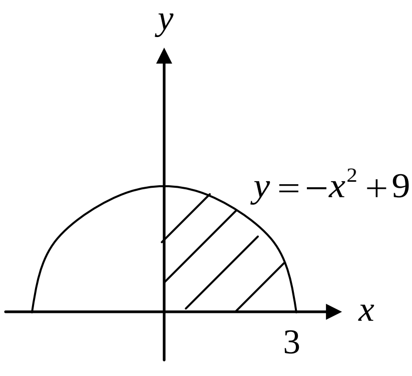


Figure 6.3

$$\text{Area, } A = \iint_R 1\,dA$$

$$= \int_0^3 \int_0^{9-x^2} 1\,dy\,dx$$

$$= \int_0^3 \left[y \right]_0^{9-x^2} dx$$

$$= \int_0^3 \left[\quad \right] dx$$

$$= \left(9x - \frac{x^3}{3} \right)_0^3$$

$$= 27 - 9$$

$$= 18 \text{ unit}^2$$

Question 6

By using the double integral, find the volume of the solid bounded by these graphs:

$x^2 + z^2 = 16, y = 2x, y = 0, z = 0$

<u>Solution</u>

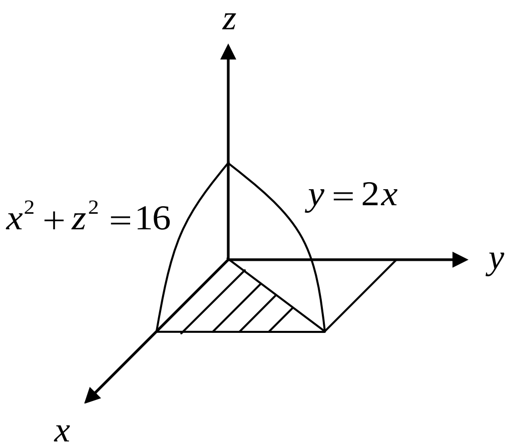


Figure 6.4 (a)

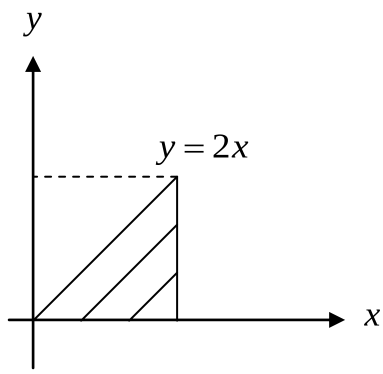


Figure 6.4 (b)

$$
\begin{aligned}
\text{Volume}, V &= \iint_R f(x,y)\,dA \\
&= \iint_R z\,dA \\
&= \int_0^4 \int_0^{2x} \sqrt{16-x^2}\,dy\,dx \\
&= \int_0^4 \left[y\sqrt{16-x^2} \right]_0^{2x} dx \\
&= \int_0^4 \left[2x\sqrt{16-x^2} \right] dx
\end{aligned}
$$

Substitution Method:

$u = 16 - x^2$ $\qquad \dfrac{du}{dx} = -2x$

$dy = -\dfrac{du}{2x}$

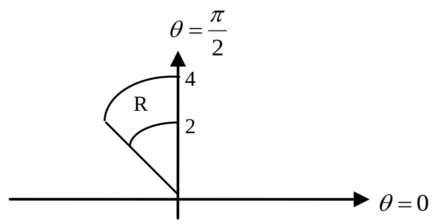
$$
\begin{aligned}
&= -\int_0^4 \left[2x\sqrt{u} \right] \frac{du}{2x} \qquad \text{- simplify } 2x \\
&= -\int_0^4 \left[u^{\frac{1}{2}} \right] du \\
&= -\left[\frac{2}{3} u^{\frac{3}{2}} \right]_0^4 \\
&= -\left[\frac{2}{3} \left(16 - x^2\right)^{\frac{3}{2}} \right]_0^4 \qquad \text{- substitude } u = 16 - x^2 \\
&= -\left[\frac{2}{3} \left(0 - 16^{\frac{3}{2}} \right) \right] \\
&= -\frac{2}{3}(-64) \\
&= 42\frac{2}{3} \ \text{unit}^3
\end{aligned}
$$

Question 7

Sketch the region bounded by the graphs below and find the area of the region by using double integral in polar coordinates :

$$r=\theta, r=4, \theta=\frac{\pi}{2}, \theta=\frac{2\pi}{3}$$

Solution :



$$\text{Area, } A=\iint_R 1\,dA=\int_{\pi/2}^{2\pi/3}\int_2^4 1\,r\,dr\,d\theta$$

$$\int_{\pi/2}^{2\pi/3}\int_2^4 1\,r\,dr\,d\theta=\int_{\pi/2}^{2\pi/3}\left[\frac{r^2}{2}\right]_2^4 d\theta$$

$$=\int_{\pi/2}^{2\pi/3}(8-2)\,d\theta$$

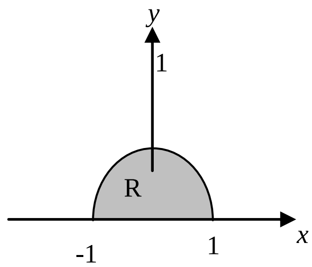
$$=\left[6\theta\right]_{\pi/2}^{2\pi/3}=6\left[\frac{2\pi}{3}-\frac{\pi}{2}\right]=\pi \text{ unit}^2$$

Question 8

By changing the integrand from Cartesian to polar coordinates , evaluate :

(a) $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$

Solution :



$$y=\sqrt{1-x^2}\Rightarrow y^2=1-x^2\Rightarrow x^2+y^2=1.$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx=\int_0^{\pi}\int_0^1 e^{-r^2}\,r\,dr\,d\theta \Rightarrow \text{Substitution Method; } u=r^2, du=2r\,dr$$

$$=\iint \frac{1}{2}e^{-u}\,du\,d\theta$$

$$=\int\left[-\frac{e^{-u}}{2}\right]d\theta$$

$$= \int_0^{\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^1 d\theta$$

$$= \int_0^{\pi} \left(-\frac{e^{-1}}{2} + \frac{1}{2} \right) d\theta$$

$$= \left(-\frac{e^{-1}}{2} + \frac{1}{2} \right) [\theta]_0^{\pi} = \left(-\frac{e^{-1}}{2} + \frac{1}{2} \right) \pi = \frac{\pi}{2}\left(1 - e^{-1}\right)$$

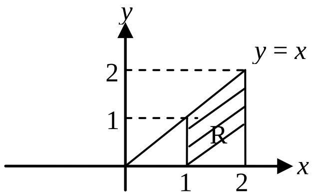
(b) $\int_1^2 \int_0^x \frac{1}{\sqrt{x^2 + y^2}} \, dydx$

<u>Solution :</u>

$x = r\cos\theta$

When $x = 1 \Rightarrow 1 = r\cos\theta \Rightarrow r = \frac{1}{\cos\theta}$

When $x = 2 \Rightarrow 2 = r\cos\theta \Rightarrow r = \frac{2}{\cos\theta}$



$$\int_1^2 \int_0^x \frac{1}{\sqrt{x^2 + y^2}} \, dydx = \int_0^{\pi/4} \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} \frac{1}{r} r \, dr d\theta$$

$$= \int_0^{\pi/4} \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} 1 \, dr d\theta$$

$$= \int_0^{\pi/4} [r]_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} d\theta$$

$$= \int_0^{\pi/4} \left(\frac{2}{\cos\theta} - \frac{1}{\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/4} \left(\frac{1}{\cos\theta} \right) d\theta$$

$$= \int_0^{\pi/4} (\sec\theta) d\theta$$

$$= [1\ln|\sec\theta + \tan\theta|]_0^{\pi/4}$$

$$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln|1| = \ln\left|\sqrt{2} + 1\right|$$

Question 9

Evaluate the iterated integral below :

$$\int_0^1 \int_{-z}^{z} \int_0^{x+z} y\,dy\,dx\,dz$$

Solution :

$$\int_0^1 \int_{-z}^{z} \int_0^{x+z} y\,dy\,dx\,dz = \int_0^1 \int_{-z}^{z} \left[\frac{y^2}{2} \right]_0^{x+z} dx\,dz$$

$$= \int_0^1 \int_{-z}^{z} \left[\frac{(x+z)^2}{2} \right]_0^{x+z} dx\,dz$$

$$= \int_0^1 \left[\frac{1}{2} \frac{(x+z)^3}{3} \right]_{-z}^{z} dz$$

$$= \frac{1}{6} \int_0^1 \left[(x+z)^3 \right]_{-z}^{z} dz$$

$$= \frac{1}{6} \int_0^1 (2z)^3 dz$$

$$= \frac{1}{6} \left[\frac{(2z)^4}{4 \cdot 2} \right]_0^1 = \frac{1}{48}[16] = \frac{1}{3}$$

Question 10

Calculate $\iiint_G ye^{2x}dV$ given that the region G is $G=\{(x,y,z):0\le x\le 1,1\le y\le 2,0\le z\le 4\}$:

Solution :

$$\int_0^4\int_1^2\int_0^1 ye^{2x}dxdydz=\int_0^4\int_1^2\left[\frac{ye^{2x}}{2}\right]_0^1 dydz$$

$$=\frac{1}{2}\int_0^4\int_1^2\left(ye^2-1\right)dydz$$

$$=\frac{1}{2}\int_0^4\left[\frac{y^2e^2}{2}-y\right]_1^2 dz$$

$$=\frac{1}{2}\int_0^4\left[\left(\frac{4e^2}{2}-2\right)-\left(\frac{e^2}{2}-1\right)\right]dz$$

$$=\frac{1}{4}\int_0^4\left(3e^2-3\right)dz$$

$$=\frac{3}{4}\left(e^2-1\right)[z]_0^4=\frac{3}{4}\left(e^2-1\right)4=3\left(e^2-1\right)$$

Question 11

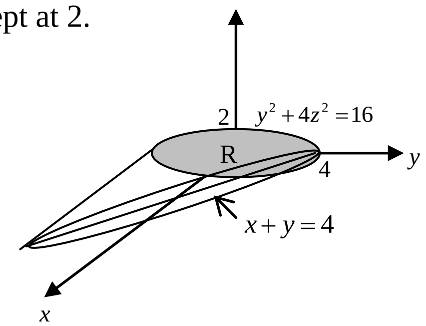
Using triple integral, find the volume of the solid bounded by the planes given below :

(a) Cylinder $y^2+4z^2=16$ and planes $x = 0$, $x + y = 0$.

Solution:

$$y^2+4z^2=16\Rightarrow\frac{y^2}{16}+\frac{z^2}{4}=1$$

Ellipse at the yz-plane, y-intercept at 4 and x-intercept at 2.



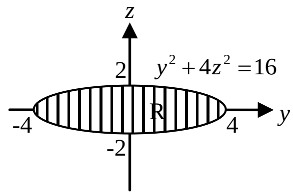
y-limit : $y = -4 \rightarrow y = 4$

z-limit : $z=-\sqrt{\dfrac{16-y^2}{4}}\ \rightarrow\ z=\sqrt{\dfrac{16-y^2}{4}}$

x-limit : $x = 0 \rightarrow x = 4 - y$.

$$V=\iiint_G 1dV=\int_{-4}^{4}\int_{-\sqrt{\frac{16-y^2}{4}}}^{\sqrt{\frac{16-y^2}{4}}}\int_0^{4-y}1dxdzdy$$

$$= \int_{-4}^{4} \int_{-\sqrt{\frac{16-y^2}{4}}}^{\sqrt{\frac{16-y^2}{4}}} [x]_0^{4-y} \, dz dy$$



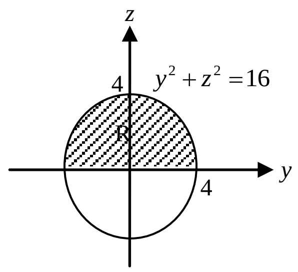
$$= \int_{-4}^{4} \int_{-\sqrt{\frac{16-y^2}{4}}}^{\sqrt{\frac{16-y^2}{4}}} [4-y] dz dy$$

$$= \int_{-4}^{4} [4z - yz]_{-\sqrt{\frac{16-y^2}{4}}}^{\sqrt{\frac{16-y^2}{4}}} dy$$

$$= \int_{-4}^{4} \left[4\sqrt{\frac{16-y^2}{4}} - y\sqrt{\frac{16-y^2}{4}} \right] - \left[-4\sqrt{\frac{16-y^2}{4}} + y\sqrt{\frac{16-y^2}{4}} \right] dy$$

$$= \int_{-4}^{4} \left[4\sqrt{16-y^2} - y\sqrt{16-y^2} \right] dy$$

$$= \underbrace{4\int_{-4}^{4} \sqrt{16-y^2}\, dy}_{(A)} - \underbrace{\int_{-4}^{4} y\sqrt{16-y^2}\, dy}_{(B)}$$



$$A = 4\underbrace{\int_{-4}^{4} \sqrt{16-y^2}\, dy}_{\frac{1}{2}\text{Area of a circle with radius } 4}$$

$$= 4\frac{1}{2}\pi(4)^2 = 32\pi$$

Or using the substitution of $y = 4\sin\theta \Rightarrow \dfrac{dy}{d\theta} = 4\cos\theta$

$$A = 4\int_{-4}^{4} \sqrt{16-y^2}\, dy = 4\int_{0}^{\pi} \sqrt{16-16\sin^2\theta} \cdot 4\cos\theta d\theta$$

$$= 64\int_0^{\pi}\sqrt{1-\sin^2\theta}\cos\theta d\theta$$

$$= 64\int_0^{\pi}\cos^2\theta d\theta$$

$$= 64\int_0^{\pi}\left[\frac{\cos 2\theta+1}{2}\right]d\theta$$

$$= 32\left[-\frac{\sin 2\theta}{2}+\theta\right]_0^{\pi}$$

$$= 32\left[\left(-\frac{\sin 2\pi}{2}+\pi\right)-0\right]$$

$$= 32\pi$$

$B = \int_{-4}^{4} y\sqrt{16-y^2}\,dy \rightarrow$ Substitution method: $u = 16-y^2 \Rightarrow \frac{du}{dy} = -2y$

$$= \int -\frac{1}{2}u^{1/2}du$$

$$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{-4}^{4}$$

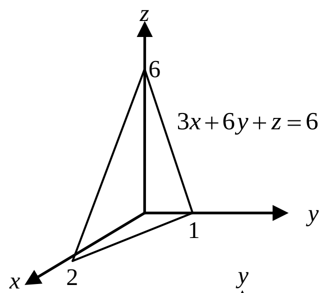
$$= -\frac{1}{3}\left[\left(16-y^2\right)\right]_{-4}^{4}$$

$$= -\frac{1}{3}(0-0) = 0$$

Therefore,

$V = \iiint_G 1dV = A + B = 32\pi + 0 = 32\pi \text{ unit}^3$

(b) $x = 0$, $y = 0$, $z = 0$, $3x + 6y + z = 6$.



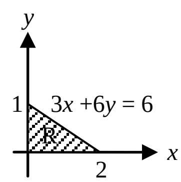
<u>Solution :</u>

When x=0, y=0 $\quad z = 6$. $\quad (0,0,6)$

When x=0, z=0 $\quad 6y = 6 \rightarrow y = 1$. $\quad (0,1,0)$

When y=0, z=0 $\quad 3x = 6 \rightarrow x = 2$. $\quad (2,0,0)$

$$V = \iiint_G 1dV = \int_0^2\int_0^{-\frac{1}{2}x+1}\int_0^{6-3x-6y} 1dzdydx$$



$$= \int_0^2 \int_0^{-\frac{1}{2}x+1} [z]_0^{6-3x-6y} dydx$$

$$= \int_0^2 \int_0^{-\frac{1}{2}x+1} (6-3x-6y) dydx$$

$$= \int_0^2 \left[6y - 3xy - 3y^2\right]_0^{-\frac{1}{2}x+1} dx$$

$$= \int_0^2 \left[6\left(-\frac{1}{2}x+1\right) - 3x\left(-\frac{1}{2}x+1\right) - 3\left(-\frac{1}{2}x+1\right)^2 - 0\right] dx$$

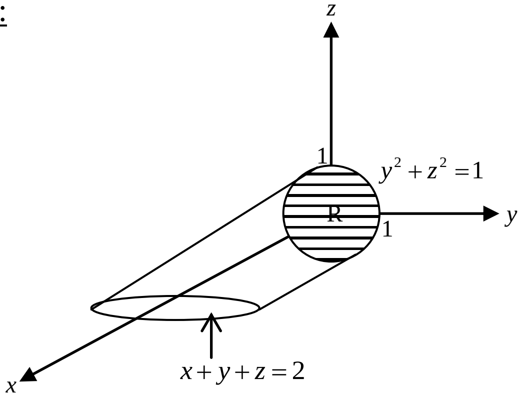
$$= \int_0^2 \left[-3x + 6 + \frac{3}{2}x^2 - 3x - \frac{3}{4}x^2 + 3x - 3\right] dx$$

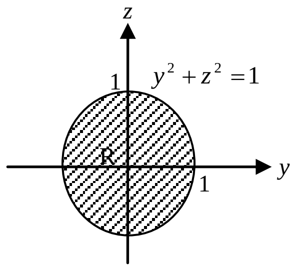
$$= \int_0^2 \left[\frac{3}{4}x^2 - 3x + 3\right] dx$$

$$= \left[\frac{3}{12}x^3 - \frac{3}{3}x^2 + 3x\right]_0^2 = 2 - 6 + 6 = 2 \text{ unit}^3$$

(c) $y^2 + z^2 = 1, x + y + z = 2, x = 0$

Solution :





$$V = \iiint_G 1dV = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{2-y-z} 1dxdzdy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} [x]_0^{2-y-z} \, dzdy$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} [2-y-z] dzdy$$

$$= \int_{-1}^{1} \left[2z - yz - \frac{z^2}{2} \right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= \int_{-1}^{1} \left[\begin{array}{l} \left\{ 2\left(\sqrt{1-y^2}\right) - y\left(\sqrt{1-y^2}\right) - \frac{\left(\sqrt{1-y^2}\right)^2}{2} \right\} \\ - \left\{ 2\left(-\sqrt{1-y^2}\right) - y\left(-\sqrt{1-y^2}\right) - \frac{\left(-\sqrt{1-y^2}\right)^2}{2} \right\} \end{array} \right] dy$$

$$= \int_{-1}^{1} \left[4\sqrt{1-y^2} - 2y\sqrt{1-y^2} \right] dy$$

$$= \underbrace{4\int_{-1}^{1} \sqrt{1-y^2} \, dy}_{(A)} - \underbrace{2\int_{-1}^{1} y\sqrt{1-y^2} \, dy}_{(B)}$$

$$A = 4 \underbrace{\int_{-1}^{1} \sqrt{1-y^2} \, dy}_{\substack{\frac{1}{2} \text{ Area of a circle} \\ \text{with radius} 1}}$$

$$= 4 \frac{1}{2} \pi (1)^2 = 2\pi$$

$B = \int_{-1}^{1} y\sqrt{1-y^2} \, dy \rightarrow$ Substitution method: $u = 1 - y^2 \Rightarrow \frac{du}{dy} = -2y$

$$= \int -\frac{1}{2} u^{1/2} du$$

$$= -\frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{-1}^{1}$$

$$= -\frac{1}{3} \left[\left(1 - y^2\right) \right]_{-1}^{1}$$

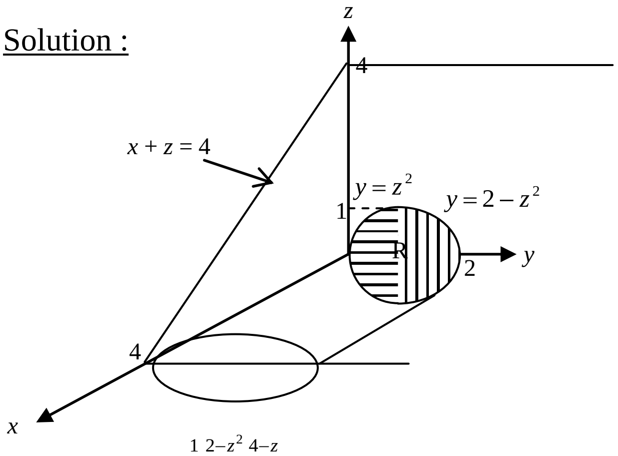
$$= -\frac{1}{3} (0 - 0) = 0$$

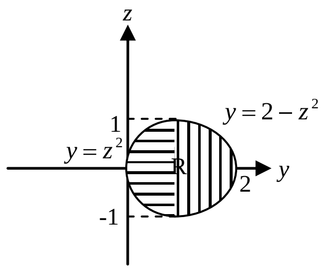
Therefore,

$$V = \iiint_G 1dV = A + B = 2\pi + 0 = 2\pi \text{ unit}^3$$

(d) $y = 2 - z^2, y = z^2, x + z = 4, x = 0$

Solution :





$$V = \iiint_G 1dV = \int_{-1}^{1}\int_{z^2}^{2-z^2}\int_{0}^{4-z} 1dxdydz$$

$$= \int_{-1}^{1}\int_{z^2}^{2-z^2} [x]_0^{4-z}\, dydz$$

$$= \int_{-1}^{1}\int_{z^2}^{2-z^2} [4 - z]dydz$$

$$= \int_{-1}^{1} [4y - zy]_{z^2}^{2-z^2}\, dz$$

$$= \int_{-1}^{1} \left[4(2 - z^2) - z(2 - z^2) - 4(z^2) + z(z^2)\right]dz$$

$$= \int_{-1}^{1} \left[8 - 4z^2 - 2z + z^3 - 4z^2 + z^3\right]dy$$

$$= \int_{-1}^{1} \left[8 - 8z^2 + 2z^3 - 2z\right]dy$$

$$= \left[8z - \frac{8}{3}z^3 + \frac{2}{4}z^4 - \frac{2}{2}z^2\right]_{-1}^{1}$$

$$= \left(8 - \frac{8}{3} + \frac{1}{2} - 1\right) - \left(-8 + \frac{8}{3} + \frac{1}{2} - 1\right)$$

$$= 10\frac{2}{3} \text{ unit}^3$$

$$A = 4\underbrace{\int_{-1}^{1}\sqrt{1-y^2}\,dy}_{\frac{1}{2}\text{ Area of a circle with radius1}}$$

$$= 4\frac{1}{2}\pi(1)^2 = 2\pi$$

$$B = \int_{-1}^{1} y\sqrt{1-y^2}\,dy$$ →Substitution method: $u = 1-y^2 \Rightarrow \frac{du}{dy} = -2y$

$$= \int -\frac{1}{2}u^{1/2}du$$

$$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{-1}^{1}$$

$$= -\frac{1}{3}\left[\left(1-y^2\right)\right]_{-1}^{1}$$

$$= -\frac{1}{3}(0-0) = 0$$

Therefore,

$$V = \iiint_G 1dV = A + B = 2\pi + 0 = 2\pi \text{ unit}^3$$

Question 12

Evaluate the iterated integral below:

(a) $\int_0^1\int_0^2\int_0^2 ydxdydz$

Solution:

$$\int_0^1\int_0^2\int_0^2 ydxdydz = \int_0^1\int_0^2 [yx]_0^2 dydz$$

$$= \int_0^1\int_0^2 2ydydz$$

$$= \int_0^1 \left[y^2\right]_0^2 dz$$

$$= \int_0^1 4dz$$

$$= [4z]_0^1 = 4$$

(b) $$\int_0^1\int_1^2\int_0^1 e^x \, dydzdx$$

Solution:

$$\int_0^1\int_1^2\int_0^1 e^x \, dydzdx = \int_0^1\int_1^2 \left[ye^x\right]_0^1 dzdx$$
$$= \int_0^1\int_1^2 e^x \, dzdx$$
$$= \int_0^1 \left[ze^x\right]_1^2 dzdx$$
$$= \int_0^1 (2e^x - e^x)dx$$
$$= \left[e^x\right]_0^1$$
$$= e^1 - e^0 = e^1 - 1$$

(c) $$\int_{-1}^1\int_1^2\int_0^1 (xy - z^2)dzdydx$$

Solution:

$$\int_{-1}^1\int_1^2\int_0^1 \left(xy - z^2\right)dzdydx = \int_{-1}^1\int_1^2 \left[xyz - \frac{z^3}{3}\right]_0^1 dydx$$
$$= \int_{-1}^1\int_1^2 (xy - \frac{1}{3})dydx$$
$$= \int_{-1}^1 \left[\frac{xy^2}{2} - \frac{y}{3}\right]_1^2 dx$$
$$= \int_{-1}^1 \left\{\left(2x - \frac{2}{3}\right) - \left(\frac{x}{2} - \frac{1}{3}\right)\right\}dx$$
$$= \int_{-1}^1 \left(\frac{3x}{2} - \frac{1}{3}\right)dx$$
$$= \left[\frac{3x^2}{4} - \frac{x}{3}\right]_{-1}^1$$
$$= \left(3\left(\frac{1^2}{4}\right) - \frac{1}{3}\right) - \left(3\left(\frac{(-1)^2}{4}\right) - \frac{-1}{3}\right) = -\frac{2}{3}$$

Question 13
Evaluate the iterated integral below:

(a)
$$\int_0^2\int_{-z}^{z}\int_0^{x+z} y\,dy\,dx\,dz$$

Solution:

$$\int_0^2\int_{-z}^{z}\int_0^{x+z} y\,dy\,dx\,dz=\int_0^2\int_{-z}^{z}\left[\frac{y^2}{2}\right]_0^{x+z}dx\,dz$$

$$=\int_0^2\int_{-z}^{z}\frac{(x+z)^2}{2}dx\,dz=\int_0^2\int_{-z}^{z}\frac{x^2+2xz+z^2}{2}dx\,dz$$

$$=\frac{1}{2}\int_0^2\left[\frac{x^3}{3}+x^2z+xz^2\right]_{-z}^{z}dz$$

$$=\frac{1}{2}\int_0^2\left\{\left(\frac{z^3}{3}+z^3+z^3\right)-\left(\frac{(-z)^3}{3}+(-z)^2z+(-z)z^2\right)\right\}dz$$

$$=\frac{1}{2}\int_0^2\frac{8}{3}z^3dz$$

$$=\frac{4}{3}\left[\frac{z^4}{4}\right]_0^2=\frac{4}{3}\left(\frac{16}{4}\right)=\frac{16}{3}$$

(b)
$$\int_0^1\int_{-1}^{y^2}\int_1^{z} yz\,dx\,dz\,dy$$

Solution:

$$\int_0^1\int_{-1}^{y^2}\int_1^{z} yz\,dx\,dz\,dy=\int_0^1\int_{-1}^{y^2}\left[yzx\right]_1^z dz\,dy$$

$$=\int_0^1\int_{-1}^{y^2}(yz^2-yz)\,dz\,dy$$

$$=\int_0^1\left[\frac{yz^3}{3}-\frac{yz^2}{2}\right]_{-1}^{y^2}dy$$

$$=\int_0^1\left\{\left(\frac{y^7}{3}-\frac{y^5}{2}\right)-\left(\frac{y(-1)^3}{3}-\frac{y(-1)^2}{2}\right)\right\}dy$$

$$=\int_0^1\left(\frac{y^7}{3}-\frac{y^5}{2}+\frac{5y}{6}\right)dy$$

$$=\left[\frac{y^8}{24}-\frac{y^6}{12}+\frac{5y^2}{12}\right]_0^1=\frac{7}{24}$$

Question 14

Calculate $\iiint_G ye^{2x}dV$ given that the region G is defined as:

$G = \{(x, y, z): 0 \le x \le 1, 1 \le y \le 2, 0 \le z \le 4\}$

Solution:

$$\int_0^1\int_1^2\int_0^4 ye^{2x}dzdydx = \int_0^1\int_1^2 \left[zye^{2x}\right]_0^4 dydx$$

$$= \int_0^1\int_1^2 4ye^{2x}dydx$$

$$= \int_0^1 \left[2y^2e^{2x}\right]_1^2 dx$$

$$= \int_0^1 \left(8e^{2x} - 2e^{2x}\right)dx$$

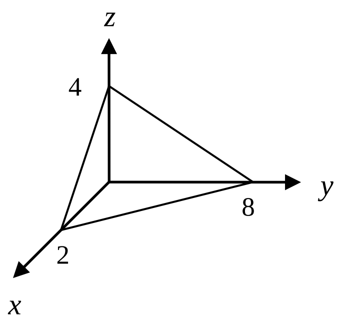
$$= \left[\frac{6e^{2x}}{2}\right]_0^1 = 3e^2 - 3e^0 = 3(e^2 - 1)$$

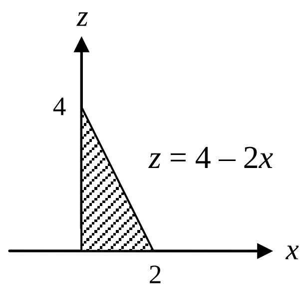
Question 15

Using triple integral, find the volume of the solid bounded by the planes given below:

(a) $x=0, y=0, z=0, 4x+y+2z=8$

Solution:





$$\text{Volume, } V = \iiint_G dV = \iint_R \left[\int_0^{8-4x-2z} dy\right] dA$$

$$= \int_0^2 \int_0^{4-2x} [y]_0^{8-4x-2z}\, dz dx$$

$$= \int_0^2 \int_0^{4-2x} (8-4x-2z)\, dz dx$$

$$= \int_0^2 \left[8z - 4xz - z^2\right]_0^{4-2x} dx$$

$$= \int_0^2 \left\{8(4-2x) - 4x(4-2x) - (4-2x)^2\right\} dx$$

$$= \int_0^2 \left(16 - 16x + 4x^2\right) dx$$

$$= \left[16x - 8x^2 + \frac{4x^3}{3}\right]_0^2$$

$$= 16(2) - 8(2^2) + \frac{4(2^3)}{3} - 0 = \frac{32}{3}\,\text{unit}^3$$

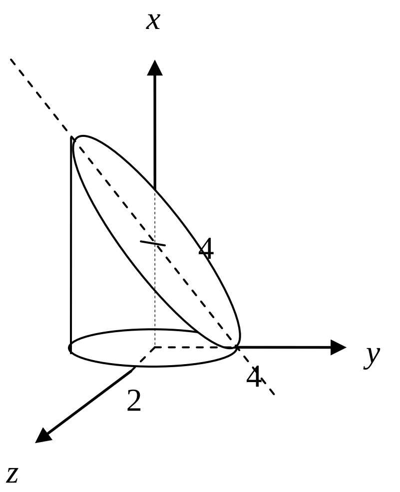
(b) Cylinder $y^2 + 4z^2 = 16$ and planes $x = 0$, $x + y = 4$

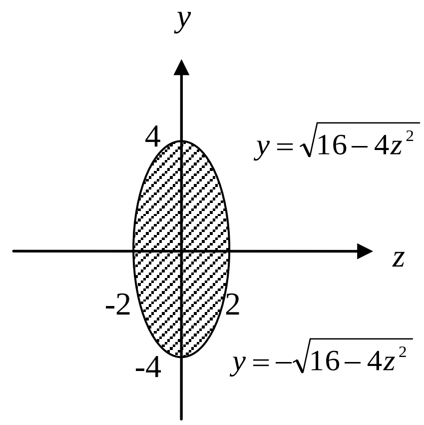
Solution:

$$y^2 + 4z^2 = 16$$

$$\frac{y^2}{16} + \frac{z^2}{4} = 1$$

$$\frac{y^2}{4^2} + \frac{z^2}{2^2} = 1$$





$$\text{Volume, } V=\iiint_G dV=\iint_R\left[\int_0^{4-y}dx\right]dA$$

$$=\int_{-2}^{2}\int_{-\sqrt{16-4z^2}}^{\sqrt{16-4z^2}}\left[x\right]_0^{4-y}dydz$$

$$=\int_{-2}^{2}\int_{-\sqrt{16-4z^2}}^{\sqrt{16-4z^2}}(4-y)dydz$$

$$=\int_{-2}^{2}\left[4y-\frac{y^2}{2}\right]_{-\sqrt{16-4z^2}}^{\sqrt{16-4z^2}}dz$$

$$=\int_{-2}^{2}\left\{\left(4(\sqrt{16-4z^2})-\frac{(\sqrt{16-4z^2})^2}{2}\right)-\left(4(-\sqrt{16-4z^2})-\frac{(-\sqrt{16-4z^2})^2}{2}\right)\right\}dz$$

$$=\int_{-2}^{2}\left\{\left(4(\sqrt{16-4z^2})-\frac{16-4z^2}{2}+4\sqrt{16-4z^2}+\frac{16-4z^2}{2}\right)\right\}dz$$

$$=\int_{-2}^{2}8\sqrt{16-4z^2}\,dz=8\int_{-2}^{2}\sqrt{4(4-z^2)}dz=8\int_{-2}^{2}2\sqrt{4-z^2}\,dz$$

$$=16\underbrace{\int_{-2}^{2}\sqrt{4-z^2}\,dz}$$

Half of area of a circle (semicircle) with radius 2
i.e.

Let

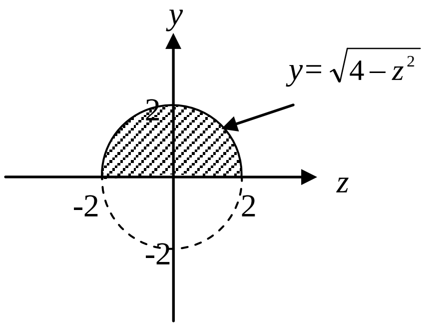
$$\int_{-2}^{2}ydz$$

where

$$y=\sqrt{4-z^2}$$

$$y^2=4-z^2$$

$$y^2+z^2=4 \quad \Rightarrow \text{circle with radius 2}$$

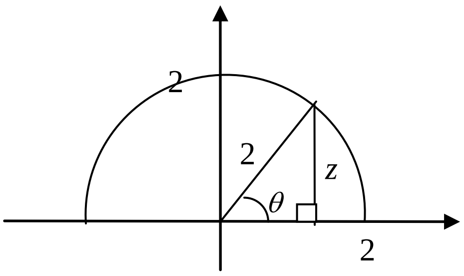


Thus, area of the shaded region (i.e semicircle) is

$$=\frac{1}{2}\pi(2)^2$$
$$=2\pi$$

$$=16\int_{-2}^{2}\sqrt{4-z^2}\,dz$$
$$=16(2\pi)$$
$$=32\pi \text{ unit}^3$$

<u>Alternative method</u>:



$$\sin\theta=\frac{z}{2}$$
$$z=2\sin\theta$$
$$\frac{dz}{d\theta}=2\cos\theta$$
$$dz=2\cos\theta d\theta$$

For

$$\int_{-2}^{2}\sqrt{4-z^2}\,dz=\int_{z=-2}^{z=2}\sqrt{4-(2\sin\theta)^2}\,2\cos\theta d\theta$$
$$=2\int_{z=-2}^{z=2}\cos\theta(\sqrt{4(1-\sin^2\theta)})d\theta$$
$$=2\int_{z=-2}^{z=2}\cos\theta\sqrt{4\cos^2\theta}\,d\theta$$
$$=4\int_{z=-2}^{z=2}\cos^2\theta d\theta$$
$$=4\int_{z=-2}^{z=2}\frac{\cos 2\theta+1}{2}d\theta$$

$$= 2\left[\frac{\sin 2\theta}{2} + \theta\right]_{z=-2}^{z=2}$$

$$= 2\left[\frac{\sin 2\left(\sin^{-1}\frac{z}{2}\right)}{2} + \sin^{-1}\frac{z}{2}\right]_{-2}^{2}$$

$$= 2\left\{\left(\frac{\sin 2\left(\sin^{-1}\frac{2}{2}\right)}{2} + \sin^{-1}\frac{2}{2}\right) - \left(\frac{\sin 2\left(\sin^{-1}\frac{2}{-2}\right)}{2} + \sin^{-1}\frac{2}{-2}\right)\right\}$$

$$= 2\left(\frac{\sin 2\left(\frac{\pi}{2}\right)}{2} + \frac{\pi}{2} - \frac{\sin 2\left(-\frac{\pi}{2}\right)}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= 2\left(0 + \frac{\pi}{2} - 0 + \frac{\pi}{2}\right)$$

$$= 2\pi$$

Thus, volume,

$$V = 16\int_{-2}^{2}\sqrt{4 - z^2}\,dz$$

$$= 16(2\pi)$$

$$= 32\pi \text{ unit}^3$$

Question16

Evaluate the cylindrical coordinate integrals below:

(a) $$\int_0^{\pi}\int_0^{\theta^2}\int_0^{r^2} z\sqrt{r}\,dz\,dr\,d\theta$$

Solution:

$$\int_0^{\pi}\int_0^{\theta^2}\int_0^{r^2} z\sqrt{r}\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{\theta^2}\left[\frac{z^2\sqrt{r}}{2}\right]_0^{r^2} dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\int_0^{\theta^2} r^4 r^{\frac{1}{2}}\,dr\,d\theta = \frac{1}{2}\int_0^{\pi}\int_0^{\theta^2} r^{\frac{9}{2}}\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[\frac{r^{\frac{11}{2}}}{\frac{11}{2}}\right]_0^{\theta^2} d\theta = \frac{1}{11}\int_0^{\pi}\left[r^{\frac{11}{2}}\right]_0^{\theta^2} d\theta$$

$$= \frac{1}{11}\int_0^{\pi}\left(\theta^2\right)^{\frac{11}{2}} d\theta = \frac{1}{11}\int_0^{\pi}\theta^{11}\,d\theta$$

$$= \frac{1}{11}\left[\frac{\theta^{12}}{12}\right]_0^{\pi} = \frac{\pi^{12}}{132}$$

(b) $$\int_0^{2\pi}\int_0^{1}\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$$

Solution:

$$\int_0^{2\pi}\int_0^{1}\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{1}\left[3zr\right]_r^{1/\sqrt{2-r^2}} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1}\left\{\frac{3r}{\sqrt{2-r^2}} - 3r^2\right\} dr\,d\theta$$

$$= \underbrace{\int_0^{2\pi}\int_0^{1}\frac{3r}{\sqrt{2-r^2}}\,dr\,d\theta}_{\mathbf{A}} - \underbrace{\int_0^{2\pi}\int_0^{1}3r^2\,dr\,d\theta}_{\mathbf{B}}$$

For **A**:

Use substitution

$$u = 2 - r^2$$

$$\frac{du}{dr} = -2r \Rightarrow dr = -\frac{du}{2r}$$

$$\int_0^{2\pi}\int_{r=0}^{r=1}\frac{3r}{u^{1/2}}\left(-\frac{du}{2r}\right)d\theta = -\frac{3}{2}\int_0^{2\pi}\int_{r=0}^{r=1}u^{-\frac{1}{2}}\,du\,d\theta = -\frac{3}{2}\int_0^{2\pi}\left[\frac{u^{\frac{1}{2}}}{1/2}\right]_{r=0}^{r=1} d\theta = -3\int_0^{2\pi}\left[\left(2-r^2\right)^{1/2}\right]_0^1 d\theta$$

$$= -3\int_0^{2\pi}\left\{\left(2-1^2\right)^{1/2} - (2-0)^{1/2}\right\}d\theta = -3\int_0^{2\pi}\left(1-\sqrt{2}\right)d\theta$$
$$= -3\left[\left(1-\sqrt{2}\right)\theta\right]_0^{2\pi} = -6\pi\left(1-\sqrt{2}\right)$$

For **B**:

$$\int_0^{2\pi}\int_0^1 3r^2\,dr\,d\theta = \int_0^{2\pi}\left[r^3\right]_0^1 d\theta = \int_0^{2\pi} d\theta = \left[\theta\right]_0^{2\pi} = 2\pi$$

Thus,

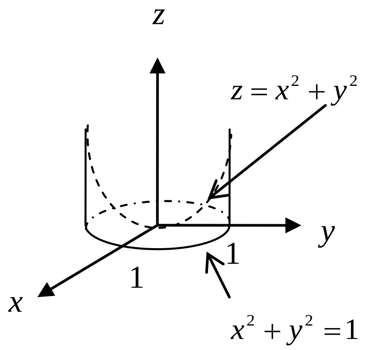
$$\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta = \mathbf{A} - \mathbf{B}$$
$$= -6\pi + 6\sqrt{2}\pi - 2\pi$$
$$= -8\pi + 6\sqrt{2}\pi = \left(6\sqrt{2}-8\right)\pi$$

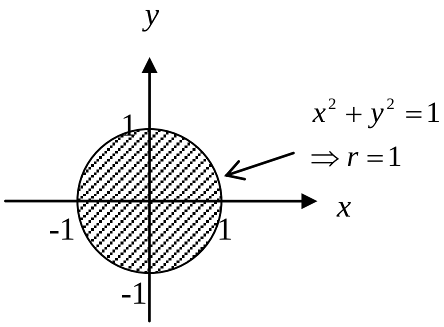
Question 17

Find the volume of the solid bounded by the surfaces below using triple integrals in cylindrical coordinate.

(a) $z = x^2 + y^2, z = 0, x^2 + y^2 = 1$

Solution:





$x^2 + y^2 = r^2 \Rightarrow z = r^2$

Volume, $V = \int_0^{2\pi}\int_0^1\int_0^{x^2+y^2} 1\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1\int_0^{r^2} r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 [rz]_0^{r^2}\, dr d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3 dr d\theta$$

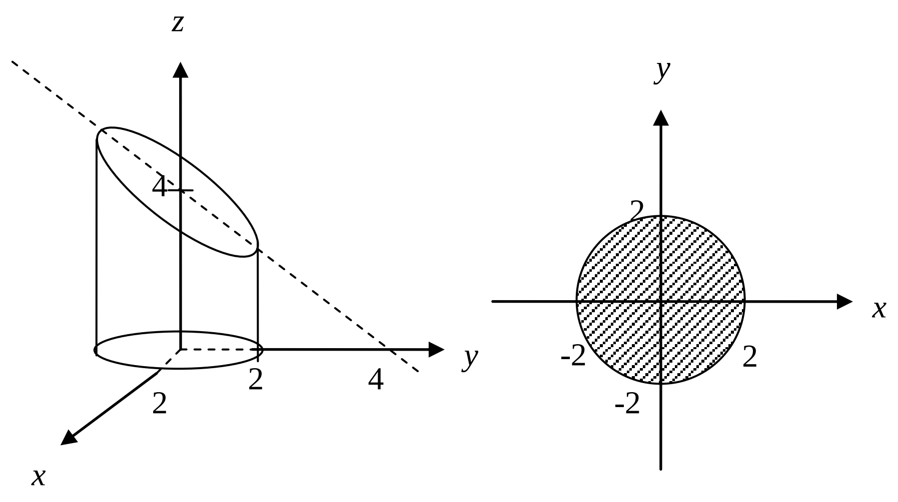
$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\frac{1}{4} d\theta$$

$$= \left[\frac{\theta}{4}\right]_0^{2\pi} = \frac{\pi}{2}\text{ unit}^3$$

(b) $y + z = 4, x^2 + y^2 = 4, z = 0$

Solution:



$$\text{Volume, } V = \int_0^{2\pi}\int_0^2\int_0^{4-y} 1 dz r dr d\theta = \int_0^{2\pi}\int_0^2\int_0^{4-r\sin\theta} r dz dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 [rz]_0^{4-r\sin\theta} dr d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(4 - r\sin\theta) dr d\theta = \int_0^{2\pi}\int_0^2 \left(4r - r^2\sin\theta\right) dr d\theta$$

$$= \int_0^{2\pi}\left[2r^2 - \frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left\{\left(2(2^2) - \frac{2^3}{3}\sin\theta\right) - 0\right\} d\theta$$

$$= \left[8\theta + \frac{8}{3}\cos\theta \right]_0^{2\pi}$$

$$= 16\pi + \frac{8}{3}\cos 2\pi - \frac{8}{3}\cos 0 = 16\pi \text{ unit}^3$$

Question 18

Evaluate the spherical coordinate integrals below:

(a) $$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Solution:

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3} \left[\rho^3 \sin\phi\right]_{\sec\phi}^{2} d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(2^3 \sin\phi - (\sec\phi)^3 \sin\phi\right) d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(8\sin\phi - \frac{\sin\phi}{\cos^3\phi}\right) d\phi \, d\theta$$

$$\boxed{\frac{\sin\phi}{\cos^3\phi} = \frac{1}{\cos^2\phi}\tan\phi = \sec^2\phi \tan\phi}$$

$$= \underbrace{\int_0^{2\pi}\int_0^{\pi/3} 8\sin\phi \, d\phi \, d\theta}_{\mathbf{A}} - \underbrace{\int_0^{2\pi}\int_0^{\pi/3} \sec^2\phi \tan\phi \, d\phi \, d\theta}_{\mathbf{B}}$$

For **A**:

$$\int_0^{2\pi}\int_0^{\pi/3} 8\sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \left[-8\cos\phi\right]_0^{\pi/3} d\theta = -8\int_0^{2\pi} \left(\cos\frac{\pi}{3} - \cos 0\right) d\theta = -8\int_0^{2\pi} -\frac{1}{2} d\theta = 4\left[\theta\right]_0^{2\pi} = 8\pi$$

For **B**:

Substitute

$$u = \tan\phi$$

$$\frac{du}{d\phi} = \sec^2\phi \Rightarrow du = \sec^2\phi \, d\phi$$

$$\int_0^{2\pi}\int_0^{\pi/3} \sec^2\phi \tan\phi \, d\phi \, d\theta = \int_0^{2\pi}\int_{\phi=0}^{\phi=\pi/3} u \, du \, d\theta = \int_0^{2\pi} \left[\frac{u^2}{2}\right]_{\phi=0}^{\phi=\pi/3} d\theta = \int_0^{2\pi} \left[\frac{\tan^2\phi}{2}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{\left(\sin\frac{\pi}{3}\right)^2}{\left(\cos\frac{\pi}{3}\right)^2} - \tan^2 0\right)d\theta = \frac{1}{2}\int_0^{2\pi}\frac{\left(\sqrt{3}/2\right)^2}{\left(1/2\right)^2}d\theta$$

$$= \int_0^{2\pi}\frac{3}{2}d\theta = \left[\frac{3}{2}\theta\right]_0^{2\pi} = 3\pi$$

Thus,

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \mathbf{A} - \mathbf{B}$$

$$= 8\pi - 3\pi$$

$$= 5\pi$$

Question 19

Find the volume of the solid bounded by the planes given below:

(a) Sphere $\rho = a$, cone $\phi = \pi/3$, and cone $\phi = 2\pi/3$

Solution:

$$\text{Volume, } V = \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\left[\frac{\rho^3}{3}\sin\phi\right]_0^a d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/3}^{2\pi/3}\frac{a^3}{3}\sin\phi \, d\phi \, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/3}^{2\pi/3} d\theta$$

$$= -\frac{a^3}{3}\int_0^{2\pi}\left(\cos\frac{2\pi}{3} - \cos\frac{\pi}{3}\right)d\theta$$

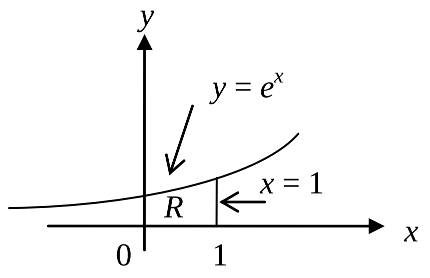
$$= -\frac{a^3}{3}\int_0^{2\pi}\left(-\frac{1}{2} - \frac{1}{2}\right)d\theta = \frac{a^3}{3}\int_0^{2\pi}1\,d\theta$$

$$= \frac{a^3}{3}\left[\theta\right]_0^{2\pi} = \frac{2a^3\pi}{3}$$

Question 20

Find the mass of the lamina region *R* bounded by the graph $y = e^x, y = 0, x = 0, x = 1$ and the density $\sigma(x, y) = y^2$.

Solution:



Mass of the lamina region *R*

$$= \iint_R \sigma(x, y)\, dA$$

$$= \int_0^1 \int_0^{e^x} y^2\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^3}{3} \right]_0^{e^x} dx$$

$$= \int_0^1 \frac{\left(e^x\right)^3}{3} dx = \int_0^1 \frac{e^{3x}}{3} dx$$

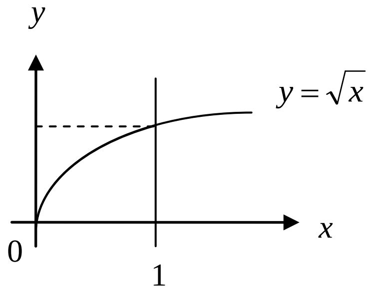
$$= \left[\frac{e^{3x}}{9} \right]_0^1 = \frac{1}{9}\left(e^3 - e^0\right) = \frac{1}{9}\left(e^3 - 1\right)$$

Question 21

Find the mass and the center of mass of lamina region *R* bounded by the given graph and density:

(a) $y = \sqrt{x}, x = 1,$ x-axis; $\sigma(x, y) = x + y$

Solution:



Mass of the lamina, $m = \int_0^1 \int_0^{\sqrt{x}} (x + y)\,dydx$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{\frac{3}{2}} + \frac{x}{2} \right) dx$$

$$= \left[\frac{x^{\frac{5}{2}}}{5/2} + \frac{x^2}{4} \right]_0^1$$

$$= \frac{2}{5}(1)^{\frac{5}{2}} + \frac{1^2}{4} = \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$$

Center of mass = $\left(\bar{x}, \bar{y}\right) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$

$$M_x = \iint_R y\sigma(x,y)\,dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} y(x + y)\,dydx = \int_0^1 \int_0^{\sqrt{x}} \left(xy + y^2\right) dydx$$

$$= \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} + \frac{x^{\frac{3}{2}}}{3} \right) dx$$

$$= \left[\frac{x^3}{6} + \frac{x^{\frac{5}{2}}}{3(5/2)} \right]_0^1$$

$$= \frac{1}{6} + \frac{2}{15} = \frac{9}{30} = \frac{3}{10}$$

$$M_y = \iint_R x\sigma(x, y)dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} x(x + y)dydx = \int_0^1 \int_0^{\sqrt{x}} (x^2 + xy)dydx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2} \right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^2}{2} \right) dx$$

$$= \left[\frac{x^{\frac{7}{2}}}{7/2} + \frac{x^3}{6} \right]_0^1$$

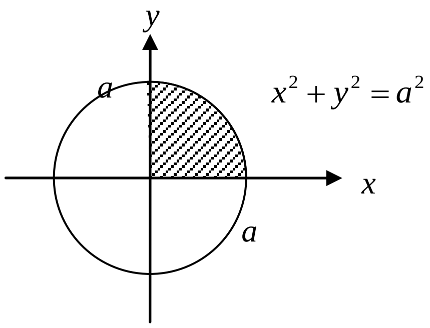
$$= \frac{2}{7} + \frac{1}{6} = \frac{19}{42}$$

$$\bar{y} = \frac{M_x}{m} = \frac{3/10}{13/20} = \frac{3}{10} \times \frac{20}{13} = \frac{6}{13}; \qquad \bar{x} = \frac{M_y}{m} = \frac{19/42}{13/20} = \frac{19}{42} \times \frac{20}{13} = \frac{190}{273}$$

Thus, center of the mass is $\left(\frac{190}{273}, \frac{6}{13} \right)$.

(b) $x^2 + y^2 = a^2$, coordinateaxes; $\sigma(x, y) = xy$

<u>Solution:</u>



Mass of the lamina, $m = \iint_R xydA$

Use double integral in polar coordinates;

Thus, $m = \int_0^{\pi/2} \int_0^a r^2 \sin\theta \cos\theta \, rdrd\theta$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4} \sin\theta \cos\theta \right]_0^a d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4} \sin\theta \cos\theta \, d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \frac{1}{2} \sin 2\theta d\theta$$

$$= \frac{a^4}{8} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{a^4}{16} \left(-\cos 2\left(\frac{\pi}{2}\right) + \cos 0 \right)$$

$$= \frac{a^4}{16} \left(-(-1) + 1 \right) = \frac{a^4}{8}$$

$$M_x = \iint_R xy^2 dA$$

$$= \int_0^{\pi/2} \int_0^a r \cos\theta \, r^2 \sin^2\theta \, r dr d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^5}{5} \cos\theta \sin^2\theta \right]_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \cos\theta \sin^2\theta d\theta$$

Use substitution

$$u = \sin\theta$$

$$du = \cos\theta d\theta$$

$$= \frac{a^5}{5} \int_{\theta=0}^{\theta=\pi/2} u^2 du$$

$$= \frac{a^5}{5} \left[\frac{u^3}{3} \right]_{\theta=0}^{\theta=\pi/2}$$

$$= \frac{a^5}{5} \left[\frac{\sin^3\theta}{3} \right]_0^{\pi/2}$$

$$= \frac{a^5}{15} \left[\left(\sin\frac{\pi}{2} \right)^3 - (\sin 0)^3 \right]$$

$$= \frac{a^5}{15}(1) = \frac{a^5}{15}$$

$$M_y = \iint_R x^2 y\,dA$$

$$= \int_0^{\pi/2}\int_0^a r^2\cos^2\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^5}{5}\cos^2\theta\sin\theta\right]_0^a d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta$$

Use substitution

$$u = \cos\theta$$

$$du = -\sin\theta\,d\theta$$

$$= \frac{a^5}{5}\int_{\theta=0}^{\theta=\pi/2} -u^2\,du$$

$$= -\frac{a^5}{5}\left[\frac{u^3}{3}\right]_{\theta=0}^{\theta=\pi/2}$$

$$= -\frac{a^5}{5}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

$$= -\frac{a^5}{15}\left[\left(\cos\frac{\pi}{2}\right)^3 - (\cos 0)^3\right]$$

$$= -\frac{a^5}{15}(0-1) = \frac{a^5}{15}$$

Thus, center of mass = $(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

$$= \left(\frac{a^5/15}{a^4/8}, \frac{a^5/15}{a^4/8}\right) = \left(\frac{a^5}{15}\times\frac{8}{a^4}, \frac{a^5}{15}\times\frac{8}{a^4}\right)$$

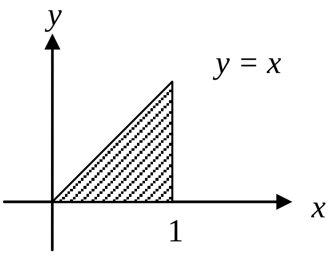
$$= \left(\frac{8a}{15}, \frac{8a}{15}\right)$$

Question 22

Find the centroid for the given regions:

(a) The triangle region enclosed by $y = x$, $x = 1$ and x – axis.

Solution:



Centroid, $(\bar{x}, \bar{y}) = \left(\dfrac{1}{\text{Area of } R} \iint_R x\,dA, \dfrac{1}{\text{Area of } R} \iint_R y\,dA \right)$

Area of R = $\frac{1}{2}(1)(1) = \frac{1}{2}$

$$\iint_R x\,dA = \int_0^1 \int_0^x x\,dy\,dx = \int_0^1 [xy]_0^x\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

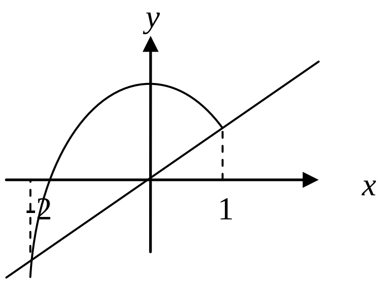
$$\iint_R y\,dA = \int_0^1 \int_0^x y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^2}{2}\,dx = \left[\frac{x^3}{6}\right]_0^1 = \frac{1}{6}$$

Thus, the centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{1}{1/2}\left(\frac{1}{3}\right), \frac{1}{1/2}\left(\frac{1}{6}\right) \right) = \left(\frac{2}{3}, \frac{1}{3} \right)$$

(b) The region bounded by $y = x$ and $y = 2 - x^2$

Solution:



$$y = x \qquad (1)$$
$$y = 2 - x^2 \qquad (2)$$

(1) = (2);

$$x = 2 - x^2$$
$$x^2 + x - 2 = 0$$
$$(x+2)(x-1) = 0$$
$$x = -2,\ x = 1$$

Area of region R

$$= \iint_R 1\,dA = \int_{-2}^{1}\int_{x}^{2-x^2} dy\,dx$$
$$= \int_{-2}^{1} [y]_x^{2-x^2}\,dx$$
$$= \int_{-2}^{1} \left(\left(2 - x^2\right) - x\right)dx$$
$$= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1}$$
$$= \left(2(1) - \frac{1^3}{3} - \frac{1^2}{2}\right) - \left(2(-2) - \frac{(-2)^3}{3} - \frac{(-2)^2}{2}\right)$$
$$= \frac{9}{2}$$

$$\iint_R x\,dA = \int_{-2}^{1}\int_{x}^{2-x^2} x\,dy\,dx$$
$$= \int_{-2}^{1} [xy]_x^{2-x^2}\,dx$$
$$= \int_{-2}^{1} \left(x(2 - x^2) - x^2\right)dx = \int_{-2}^{1} \left(2x - x^3 - x^2\right)dx$$
$$= \left[x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-2}^{1}$$
$$= \left(1 - \frac{1}{4} - \frac{1}{3}\right) - \left((-2)^2 - \frac{(-2)^4}{4} - \frac{(-2)^3}{3}\right) = -\frac{9}{4}$$

$$\iint_R y\,dA = \int_{-2}^{1}\int_{x}^{2-x^2} y\,dy\,dx$$

$$= \int_{-2}^{1}\left[\frac{y^2}{2}\right]_x^{2-x^2} dx$$

$$= \frac{1}{2}\int_{-2}^{1}\left[\left(2-x^2\right)^2 - x^2\right]dx = \frac{1}{2}\int_{-2}^{1}\left(4-4x^2+x^4-x^2\right)dx = \frac{1}{2}\int_{-2}^{1}\left(4-5x^2+x^4\right)dx$$

$$= \frac{1}{2}\left[4x - \frac{5x^3}{3} + \frac{x^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(4-\frac{5}{3}+\frac{1}{5}\right) - \left(4(-2) - \frac{5(-2)^3}{3} + \frac{(-2)^5}{5}\right)\right] = \frac{9}{5}$$

Thus, the centroid is

$$\left(\bar{x},\bar{y}\right) = \left(\frac{1}{9/2}\left(\frac{-9}{4}\right), \frac{1}{9/2}\left(\frac{9}{5}\right)\right)$$

$$= \left(\frac{2}{9}\times\frac{-9}{4}, \frac{2}{9}\times\frac{9}{5}\right) = \left(-\frac{1}{2}, \frac{2}{5}\right)$$