CHAPTER 3 APPLICATIONS OF DIFFERENTIATION

3.1 Approximate Value and Error (page 151)

$$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{f(x+\delta x) - f(x)}{\delta x} \approx f'(x)$$

or

$$f(x+\delta x) - f(x) \approx f'(x)\delta x$$

$$\boxed{f(x+\delta x) \approx f(x) + f'(x)\delta x \quad (3.2)}$$

$$\delta f(x) \approx f'(x)\delta x \quad (3.3)$$

where $\delta f(x) = f(x+\delta x) - f(x)$

Example 3.1 (page 152):

Find the approximate value of $\sqrt[3]{8.03}$.

Example 3.2 (page 152):

The radius of a circle is measured with a possible error 2%. Find the possible percentage error in calculating the area of the circle.

3.2 Rates of Change (page 155)

If y is a function of x, then $\frac{dy}{dx}$ is the change of y changes with respect to x. Example, if V represents the volume of an object and r represents the radius, V is the function in terms of r, then $\frac{dV}{dr}$ represents the rate of change of the volume with respect to radius.

Example 3.6 (page 156):

The radius, r cm of a sphere at t seconds is given by

$$r = 4 + \frac{3}{2+t} + 2t^2$$

a) Find the initial radius of the sphere.

b) Find the rate of change of r at $t = 3$.

3.2.1.1 Constant Rate of Change (page 158)

If $\frac{dr}{dt} = c$, where c is a constant value, $\frac{dr}{dt} = c$ is known as a constant rate of change. It means that for every value of t, $\frac{dr}{dt}$ is always the same. Hence for the constant rate of change,

$$\frac{dr}{dt} = \frac{\text{changing of } r}{\text{changing of } t}$$

Example 3.9 (page 158):

The radius r cm of a circle increases at a constant rate of 0.5 cm s^{-1}. If the initial radius is 3.5 cm, find the radius of the circle after 10 seconds.

3.2.2 Related Rates (page 158)

The problem involving the rate of change of several related quantities is called the related rate of change problem. In general, this problem can be solved by using the differentiation of composite functions (chain rule).

Example 3.14 (page 162):

The length of each side of a cube being heated is increasing at a constant rate of 2×10^{-3} cm/s.

a) State the total area of the surface, *A* in terms of x, the length of its side, and find $\frac{dA}{dx}$.

b) If the initial length of the sides is 10 cm, after the cube has been heated for 50 seconds calculate the length of the sides.

c) Calculate the rate of increase in the surface area at that time.

3.3 Motion Along a Line

(Omitted)

3.4 Gradient of Curve at a Point (page 176)

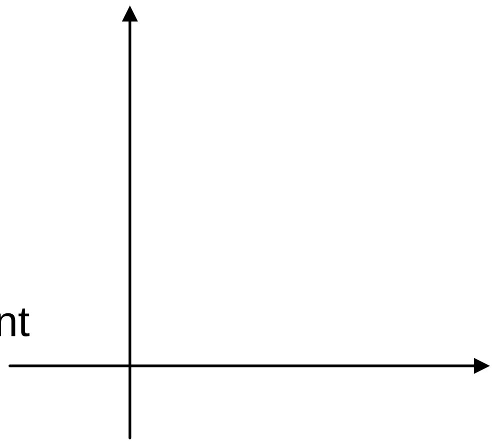
The gradient of curve at a point can be defined as the gradient of tangent line at the point of the curve.

Example 3.22 (page 176):

Find the gradient of the curve $y = x^3 + 8x - 5$ at (2, 19)

3.4.1 Equation of Tangent to a Curve

The tangent to the curve $y = f(x)$ at any point P is the straight line PA which touches the curve at point P. To find the equation of the tangent to the curve at P, we need to find the gradient of the tangent to curve at that point.



Example 3.23 (page 177):

Find the equation of tangent of the parabola $y = x^2 - 5x + 3$ at $x = 3$.

Example 3.24 (page 177):

Find the equation of tangent to the ellipse $3x^2 + 4y^2 = 48$ at (2, 3).

Example 3.25 (page 178):

The parametric equation of a curve is given by

$$x = t^2 + 1, \quad y = t^3.$$

Show that the point (5, -8) is a point on the curve and find the equation of the tangent to the curve at that point.

3.4.2 Equation of Normal to a Curve

The normal line equation to the curve $y = f(x)$ at any point P can be defined as a straight line PB that is perpendicular to the tangent PA. If the gradient of the tangent to the curve at P is m, then the gradient of the normal at P is $-\frac{1}{m}$.

(gradient of the tangent) $\times$ (gradient of the normal)= -1

Example 3.29 (page 182):

Find the equation of the normal to the curve $y = 2x^3 - 3x^2 + 5$ at (2, 9).

Exercise at home: (Tutorial 5)

Quiz 3A (page 153) : 1, 4b)

Quiz 3B (page 167) : 4, 5

Quiz 3D (page 184) : 5b), 11

Exercise 3 (page 231) : 1, 9, 16, 22

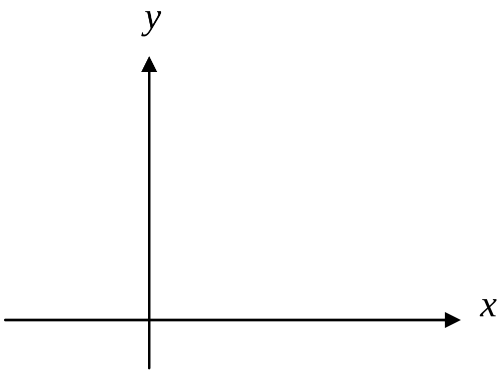
3.5 Maximum and Minimum (page 186)

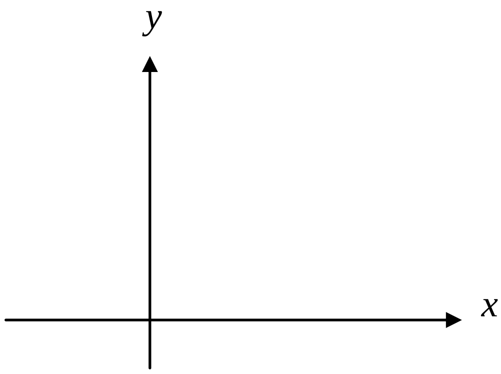
For any two values of x, such as x_1 and x_2 where $x_1 < x_2$,

(i) $y = f(x)$ is increasing if $f(x_1) < f(x_2)$

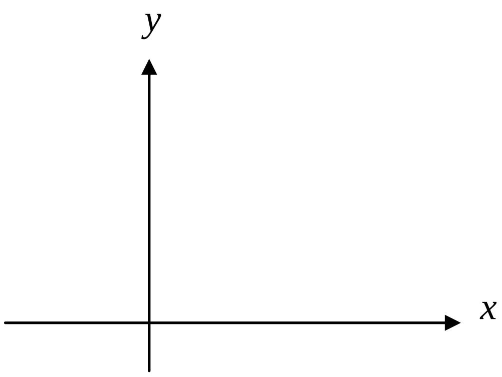
(ii) $y = f(x)$ is decreasing if $f(x_1) > f(x_2)$

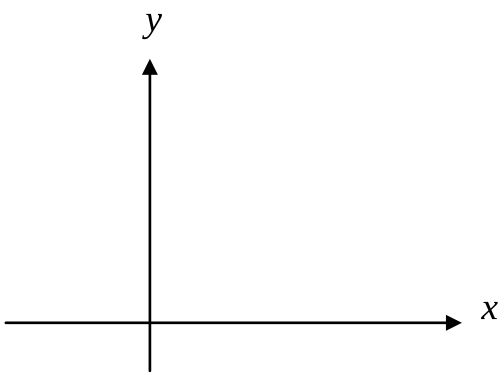
(iii) $y = f(x)$ is constant if $f(x_1) = f(x_2)$





Increasing function: positive gradient everywhere (page 186)





Decreasing function: negative gradient everywhere (page 187)

Assume that $y = f(x)$ is a continuous function and differentiable on the open interval $a < x < b$. (page 186)

(i) If $f'(x) > 0$, then $y = f(x)$ is increasing on the interval.

(ii) If $f'(x) < 0$, then $y = f(x)$ is decreasing on the interval.

$f'(x)$ is increasing, f is concave (page 188 & 193)

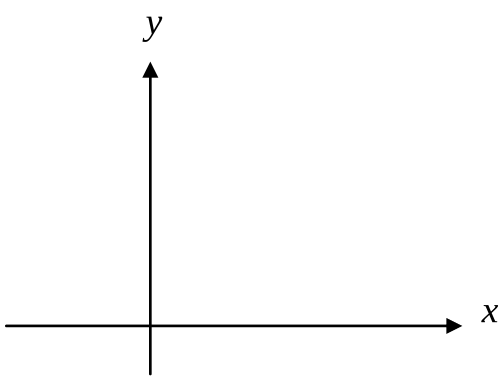
$f'(x)$ is decreasing, f is convex

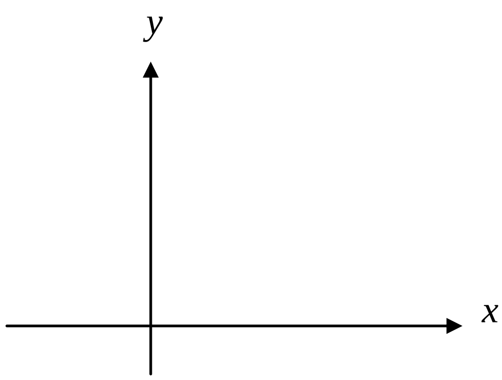
(i) If $f''(x) > 0$, then $f'(x)$ is increasing and

$y = f(x)$ is concave.

(ii) If $f''(x) < 0$, then $f'(x)$ is decreasing and

$y = f(x)$ is convex.





Definition 3.1 (Critical Points) (page 188)

A point $(c, f(c))$ of the function $f(x)$ is a critical point if $f'(c) = 0$ or if $f'(c)$ does not exist.

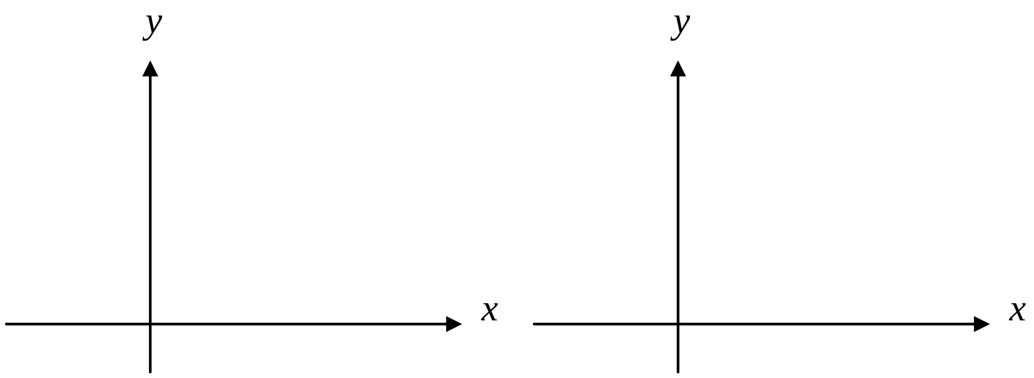
First Derivative Test (page 189)

Given that $y = f(x)$.

1. If $f'(x) = 0$, or, if $f'(x)$ does not exist, x is a critical point.

2. A critical point is a maximum point if $f'(x)$ changes sign from positive to negative as x is increasing through the critical point. The curve is convex.

3. A critical point is a minimum point if $f'(x)$ changes sign from negative to positive as x is increasing through the critical point. The curve is concave.



Example 3.33 (page 189):

Find the maximum and minimum points (if any) of the functions below.

(a) $f(x) = x^2 - 4x - 1$ (b) $f(x) = x^3 - 9x^2 + 15x - 5$

Definition 3.2 (Absolute Maximum / Minimum Values) (page 192)

Absolute maximum value — the function has the largest value in the domain. Similarly, the absolute minimum value is the smaller than all other values of $f(x)$ in the domain.

Definition 3.3 (Local Maximum / Minimum Values) (Page 192)

A local or relative maximum/minimum value of a function occurs when the function has the larger/smaller value in its neighborhood. Maximum and minimum values are called extreme values of the function.

Second Derivative Test (page 193)

Assuming that $y = f(x)$ has a critical point at $x = x_0$.

1. If $f''(x) < 0$, the graph is convex and $f(x)$ has a maximum at $x = x_0$.
2. If $f''(x) > 0$, the graph is concave and $f(x)$ has a minimum at $x = x_0$.
3. If $f''(x) = 0$, or does not exist, the second derivative test fails. We have to use the first derivative to determine the property of the extreme point at $x = x_0$.

Example:

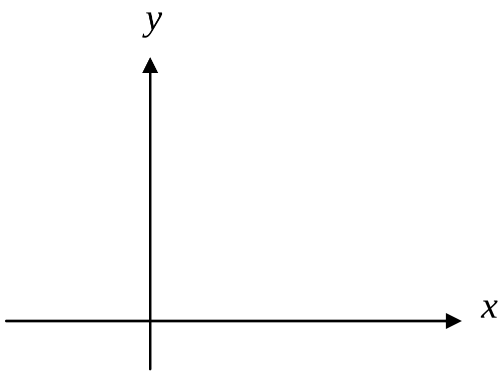
Solve Example 3.33 (b) (Page 189)

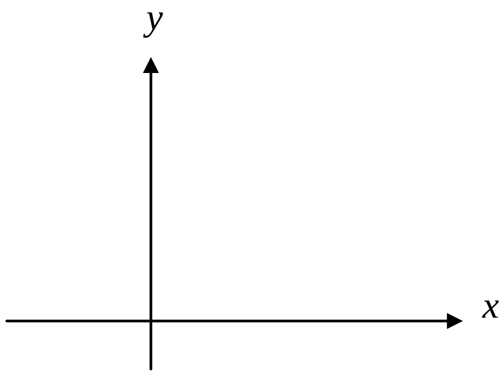
Example 3.36 (page 196):

Determine the maximum and minimum points of $f(x) = 1 - x^4$

Definition 3.4 (Point of Inflection) (page 197)

A point which separates a convex and a concave sections of a continuous function is called a point o inflection.





There are three possibilities when $f''(x_0) = 0$ at the critical point $x = x_0$. They are (page 198)

(a) maximum point (b) minimum point (c) point of inflection

(A point of inflection need not be a critical point. Only the condition of f''(x) = 0 is a necessity).

Theorem 3.1 (Point of Inflection and Sign $f''(x_0)$)

If $f''(x_0) = 0$ or if $f''(x_0)$ does not exist, and if the value of $f''(x)$ changes sign when passing through $x = x_0$, then the point $(x_0, f(x_0))$ on the curve is a point of inflection.

Example 3.37 (page 198):

Determine maximum, minimum and the point of inflection (if any) of $f(x) = (x-1)^3$.

3.6 Curve Sketching (omitted) (page 212)

3.7 L'Hospital's Rule (page 227)

Definition 3.8 (L'Hospital's Rule) (page 227)

Assume that $f(x)$ and $g(x)$ are differentiable functions in the interval (a,b) containing c except (possibly) at the point c itself. If $\frac{f(x)}{g(x)}$ has an improper form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at c and if $g'(x) \neq 0$ for $x \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)},$$

provided

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists} \qquad \text{or} \qquad \lim_{x \to c} \frac{f'(x)}{g'(x)} = \infty$$

Note (page 229):

If the first derivatives still give an undetermined form then we apply L'Hospital's once again. The repeated use of L'Hospital's rule is permitted.

Example 3.53 (page 228):

(a) $\lim_{x \to -3} \frac{x^2 + 6x + 9}{x + 3}$

(d) $\lim_{x \to 0} \frac{\sin x}{x}$

Example 3.55 (page 229):

(c) $\lim_{x \to \infty} \frac{x^2}{e^x}$

Exercise at home (Tutorial 6):

(page 208), Quiz 3E: no. 2(a), 3(a), 3(b)

(page 231), Quiz 3G: no. 1, 3, 7, 13, 15

(page 240 & 242), Exercise 3: no. 56, 62