CHAPTER THREE

FOURIER SERIES

After completing these tutorials, students should be able to:

- determine whether the given function is even, odd or neither.
- compute the Fourier Series for the given function
- find the Fourier Series for the given function
- determine the Fourier Cosine Series for the given function on the indicated interval
- determine the Fourier Sine Series for the given function on the indicated interval
- find the half-range Cosine and Sine expansions of the given function
- apply the Fourier Series in solving engineering problems

Question 1

In problem I throught III:

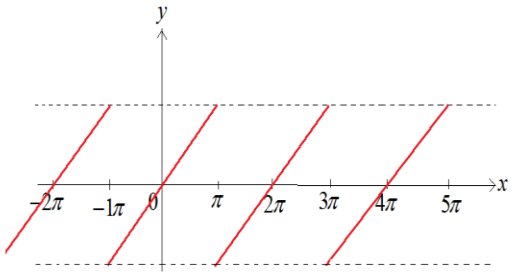
a) Sketch the graph of the given function.
b) Determine whether the given function is even, odd or neither.
c) Compute the Fourier Series for the given function.

I $f(x) = x;\ -\pi \prec x \prec \pi$

$f(x+2\pi) = f(x)$

Solution:

(a) The graph:



(b) $f(x)$ is symmetric with respect to the origin

- 0dd function
- Fourier Sine Series

(c) $$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nxdx$$

$$= \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right]$$

$$= -\frac{2}{\pi}\cos n\pi$$

$$= \begin{cases} -\frac{2}{n}(-1) & ; \quad n = 1,3,5,\ldots \\ -\frac{2}{n}(1) & ; \quad n = 2,4,6,\ldots \end{cases}$$

$$= \begin{cases} \frac{2}{n} & ; \quad n = odd \\ -\frac{2}{n} & ; \quad n = even \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\pi}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\frac{n\pi x}{\pi}$$

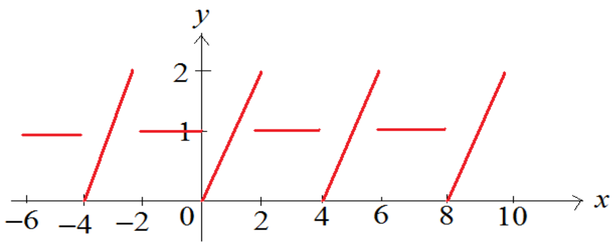
$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

II $$f(x) = \begin{cases} 1 \; ; & -2 \prec x \prec 0 \\ x \; ; & 0 \prec x \prec 2 \end{cases}$$

$$f(x+4) = f(x)$$

Solution:

(a) The graph:



(b) $f(x)$ is neither even nor an odd function.

(c) $$a_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)dx$$

$$= \frac{1}{2}\left[\int_{-2}^{n} 1dx + \int_0^2 xdx\right]$$

$$= \frac{1}{2}\left[x\Big|_{x=-2}^{0} + \frac{x^2}{2}\Big|_{x=0}^{2} \right]$$
$$= \frac{1}{2}(0+2+2-0)$$
$$= 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$
$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\left[\int_{-2}^{0} 1\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$
$$= \frac{1}{2}\left[\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \left[\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{2}\right]$$
$$= \frac{1}{2}.\frac{2}{n\pi}\left[(0-0) + \left(0 + \frac{2}{n\pi}\cos n\pi\right) - \left(0 + \frac{2}{n\pi}\right)\right]$$
$$= \frac{2}{(n\pi)^2}(\cos n\pi - 1)$$
$$= \begin{cases} \dfrac{2}{(n\pi)^2}(-1-1) \quad ; \quad n = odd \\ \dfrac{2}{(n\pi)^2}(1-1) \quad ; \quad n = even \end{cases}$$
$$= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$
$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$
$$= \frac{1}{2}\left[\int_{-2}^{0} 1\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \left[-\frac{2}{n\pi}x\cos\left(\frac{n\pi x}{2}\right) + \left(\frac{2}{n\pi}\right)^{2}\sin\frac{n\pi x}{2}\right]_{0}^{2}\right]$$

$$= \frac{1}{2}.\frac{2}{n\pi}\left[-1+\cos(-n\pi)-2\cos n\pi+0+0-0\right]$$

$$= \frac{1}{n\pi}(-1-\cos n\pi)$$

$$= \begin{cases} \dfrac{1}{n\pi}(-1-(-1)) & ; \quad n = odd \\ \dfrac{1}{n\pi}(-1-1) & ; \quad n = even \end{cases}$$

$$= \frac{1}{n\pi}\left(-1-(-1)^{n}\right) = \frac{1}{n\pi}(-1+(-1)^{n+1})$$

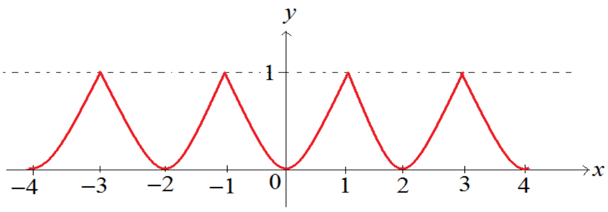
$$\therefore f(x) = 1 + \sum_{n=1}^{\infty}\left[\frac{2}{(n\pi)^{2}}\left((-1)^{2}-1\right)\cos\frac{n\pi x}{2} + \frac{1}{n\pi}(-1+(-1)^{n+1})\sin\frac{n\pi x}{2}\right]$$

III $f(t) = t^{2}$; $-1 \prec t \prec 1$

$f(t+2) = f(x)$

Solution:

(a) The graph:



(b) $f(x)$ is symmetrical with y-axis

- Even function
- Cosine Fourier Series

(c) $a_0 = \dfrac{2}{1}\displaystyle\int_0^L x^2 dx$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}(1-0)$$

$$a_n = \frac{2}{1}\int_0^1 x^2 \cos\frac{n\pi x}{1}dx$$

$$= 2\left[\frac{x^2 \sin n\pi x}{nx} + \int \frac{2x\sin n\pi x}{nx}dx\right]_0^1$$

$$= 2\left[\frac{1}{nx}x^2 \sin n\pi x - \frac{2}{nx}\left(\frac{-x\cos n\pi x}{n\pi} - \frac{\sin n\pi x}{(n\pi)^2}\right)\right]_0^1$$

$$= 2\left[\frac{2}{(n\pi)^2}\left(1\cos n\pi - 0\right)\right]$$

$$= \frac{4}{(n\pi)^2}(\cos n\pi)$$

$$= \begin{cases} \frac{4}{(n\pi)^2}(-1) & ; \quad n = 1,3,5,\ldots \\ \frac{4}{(n\pi)^2}(1) & ; \quad n = 2,4,6,\ldots \end{cases}$$

$$= \frac{4}{(n\pi)^2}(-1)^n$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^n \cos n\pi x$$

Question 2

Find the Fourier Series for the given function.

(a) $$f(x) = \begin{cases} 0 & ; \; -\pi \prec x \prec 0 \\ x^2 & ; \; 0 \le x \prec \pi \end{cases}$$

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_0^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nxdx + \int_{0}^{\pi} x^2 \cos nxdx\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{x^2 \sin nx}{n}\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi - \frac{2}{n}\left[\left[\frac{-x\cos nx}{n}\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nxdx\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi - \frac{2}{n}\left[\frac{-x\cos nx}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right]\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{n}\sin n\pi + \left(\frac{2\pi\cos n\pi}{n^2}\right) - \frac{2\sin n\pi}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi\cos n\pi}{n^2}\right)$$

$$= \frac{2\pi}{n^2}\cos n\pi$$

$$= \frac{2\pi}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nxdx + \int_{0}^{\pi} x^2 \sin nxdx\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{x^2 \cos nx}{n}\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\cos nxdx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n}\left[\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right]\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^2}\pi\sin n\pi + \frac{2}{n}\left[\frac{\cos nx}{n^2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^2}\pi\sin n\pi + \frac{2\pi\cos n\pi}{n^3} - \frac{2}{n^3}\right)$$

$$= -\frac{\pi}{n}(-1)^n + \frac{2(-1)^n}{\pi n^3} - \frac{2}{\pi n^3}$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2}(-1)^n\cos nx + \left[\frac{(-1)^{n+1}\pi}{n} + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]\right]\sin nx$$

Question 3

Determine the Fourier Cosine Series for the following functions on the indicated interval.

(a) $$f(x)=\begin{cases}0 \ ; & 0 \prec x \prec \pi \\ 1 \ ; & \pi \le x \prec 2\pi\end{cases}$$

Solution:

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(x)dx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} 0dx + \int_{\pi}^{2\pi} 1dx\right]$$
$$= \frac{1}{\pi}[x]_{\pi}^{2\pi}$$
$$= \frac{1}{\pi}(2\pi - \pi)$$
$$=1$$

$$a_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$
$$= \frac{1}{\pi}\left[\int_0^{\pi} 0\cos\frac{nx}{2}dx + \int_{\pi}^{2\pi} 1.\cos\frac{nx}{2}dx\right]$$
$$= \frac{1}{\pi}\left[\frac{2}{n}\sin\frac{nx}{2}\right]_{\pi}^{2\pi}$$
$$= \frac{1}{\pi}\left(\frac{2}{n}\sin n\pi - \frac{2}{n}\sin\frac{n\pi}{2}\right)$$
$$= \frac{1}{\pi}\left(-\frac{2}{n}\sin\frac{n\pi}{2}\right)$$

$$\therefore\ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{nx}{2}\right)$$
$$= \frac{1}{2} - \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{nx}{2}\right)$$

(b) $f(x)=e^{-x};\ 0 \prec x \prec 1$

Solution:

$$a_0 = \frac{2}{1}\int_0^1 f(x)dx$$
$$= 2\int_0^1 e^{-x}dx$$

$$= 2\left[-e^{-x}\right]_0^1$$
$$= 2(-e^{-1} + e^0)$$
$$= 2\left(1 - \frac{1}{e^1}\right)$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}dx$$
$$= 2\int_0^1 e^{-x}\cos n\pi x dx$$

$u = e^{-x}$	$v' = \cos n\pi x$
$u' = -e^{-x}$	$v = \dfrac{\sin n\pi x}{n\pi}$

$$\int_0^1 e^{-x}\cos n\pi x dx = \left[\frac{e^{-x}}{n\pi}\sin n\pi x\right]_0^1 + \int_0^1 \frac{e^{-x}}{n\pi}\sin n\pi x dx$$

$u = e^{-x}$	$v' = \sin n\pi x$
$u' = -e^{-x}$	$v = -\dfrac{\cos n\pi x}{n\pi}$

$$\int_0^1 e^{-x}\cos n\pi x dx + \int_0^1 \frac{e^{-x}}{n\pi}\cos n\pi x dx = \frac{e^{-1}}{n\pi}\sin n\pi - \frac{e^{-1}}{n\pi}\cos n\pi + \frac{1}{n\pi}$$
$$\left(1 + \frac{1}{n\pi}\right)\int_0^1 e^{-x}\cos n\pi x dx = -\frac{e^{-1}}{n\pi}\cos n\pi + \frac{1}{n\pi}$$
$$\int_0^1 e^{-x}\cos n\pi x dx = \frac{1 - e^{-1}(-1)^n}{n\pi + 1}$$

$$a_n = 2\int_0^1 e^{-x}\cos n\pi x dx$$
$$= \frac{2 - 2e^{-1}(-1)^n}{n\pi + 1}$$

$$\therefore f(x) = \left(1 - \frac{1}{e}\right) + \sum_{n=1}^{\infty}\frac{2 - 2e^{-1}(-1)^n}{n\pi + 1}\cos n\pi x$$
$$= \left(1 - \frac{1}{e}\right) + 2\sum_{n=1}^{\infty}\frac{1 - e^{-1}(-1)^n}{n\pi + 1}\cos n\pi x$$

Question 4

Determine the Fourier Sine Series for the following functions on the indicated interval.

(a) $f(x) = x + 1;\ 0 \prec x \prec 1$

Solution:

$$b_n = \frac{2}{1}\int_0^1 f(x)\sin\frac{n\pi x}{1}dx$$

$$= 2\int_0^1 (x+1)\sin n\pi x dx$$

$$= 2\left[\int_0^1 x\sin n\pi x dx + \int_0^1 \sin n\pi x dx\right]$$

$$= 2\left[\left[\frac{-x\cos n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi} + \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1\right]$$

$$= 2\left[\frac{-\cos n\pi}{n\pi} + \left[\frac{\sin n\pi x}{(n\pi)^2}\right]_0^1 - \frac{\cos n\pi}{n\pi} + \frac{1}{n\pi}\right]$$

$$= 2\left(\frac{1}{n\pi} - \frac{2\cos n\pi}{n\pi}\right)$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-2(-1)^n\right)\sin n\pi x$$

(b) $f(x) = \begin{cases} x & ;\ 0 \prec x \prec 1 \\ 2-x & ;\ 1 \le x \prec 2 \end{cases}$

Solution:

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$

$$= \int_0^1 x\sin\frac{n\pi}{2}xdx + \int_1^2 (2-x)\sin\frac{n\pi}{2}xdx$$

$$= \underbrace{\int_0^1 x\sin\frac{n\pi}{2}xdx}_{case1} + \underbrace{\int_1^2 2\sin\frac{n\pi}{2}xdx}_{case2} - \underbrace{\int_1^2 x\sin\frac{n\pi}{2}xdx}_{case3}$$

$$u = x$$

$$u' = 1$$

$$v' = \sin\frac{n\pi x}{2}$$

$$v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$$

Case 1:

$u = x$	$v' = \sin\frac{n\pi x}{2}$
$u' = 1$	$v = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}$

$$\int_0^1 x\sin\frac{n\pi}{2}xdx = \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^1 + \int_0^1\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2\left[\sin\frac{n\pi x}{2}\right]_0^1$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}$$

Case 2:

$$\int_1^2 2\sin\frac{n\pi}{2}xdx = 2\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2$$

$$= -\frac{4}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

Case 3:

$$\int_1^2 x\sin\frac{n\pi}{2}xdx = \left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_1^2 + \left(\frac{2}{n\pi}\right)^2\left[\sin\frac{n\pi x}{2}\right]_1^2$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{(n\pi)^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)$$

$$b_n = -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2} - \frac{4}{n\pi}\cos n\pi + \frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n\pi}\cos n\pi - \frac{2}{n\pi}\cos\frac{n\pi}{2}$$

$$+ \frac{4}{(n\pi)^2}\sin n\pi + \frac{4}{(n\pi)^2}\sin\frac{n\pi}{2}$$

$$= \frac{8}{(n\pi)^2}\sin\frac{n\pi}{2}$$

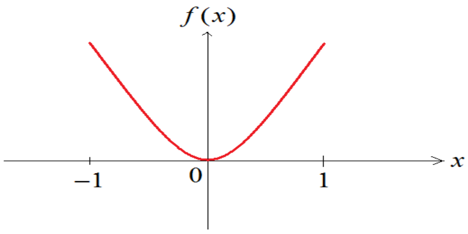
$$\therefore f(x) = \sum_{n+1}^{\infty}\left(\frac{8}{(n\pi)^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2}\right)$$

Question 5

Expand the given function in an appropriate Cosine or Sine Series.

(a) $f(x) = x^2;\ -1 \prec x \prec 1$

<u>Solution:</u>



$f(x)$ is an even function, $b_n = 0$

$$a_0 = \frac{2}{1}\int_0^1 f(x)dx$$

$$= 2\int_0^1 x^2 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{3}\right)$$

$$= \frac{2}{3}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1}dx$$

$$= 2\int_0^1 x^2 \cos n\pi x dx$$

$u = x^2$	$v' = \cos n\pi x$
$u' = 2x$	$v = \dfrac{\sin n\pi x}{n\pi}$

$$a_n = 2\left[\left[\frac{x^2 \sin n\pi x}{n\pi}\right]_0^1 - 2\int_0^1 \frac{x \sin n\pi x}{n\pi} dx\right]$$

$u = x$	$v' = \sin n\pi x$
$u' = 1$	$v = -\dfrac{\cos n\pi x}{n\pi}$

$$a_n = 2 - \left[\frac{\sin n\pi}{n\pi} - \frac{2}{n\pi}\left[\left[\frac{-x\cos n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\right]\right]$$

$$= 2\left[-\frac{2}{n\pi} + \frac{2}{n\pi}\left[\frac{-\cos n\pi}{n\pi} + \left(\frac{\sin n\pi}{(n\pi)^2}\right)_0^1\right]\right]$$

$$= 2\left[-\frac{2}{n\pi} + \frac{2\cos n\pi}{(n\pi)^2} - \frac{2\sin n\pi}{(n\pi)^3}\right]$$

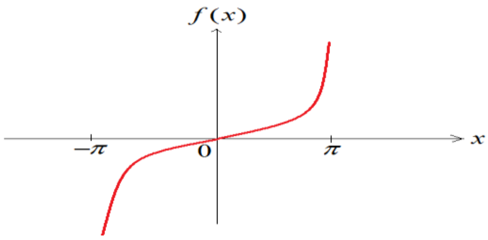
$$= 2\left(-\frac{2}{n\pi} + \frac{2(-1)^n}{(n\pi)^2}\right)$$

$$= \frac{4}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n}\right)$$

$$\therefore f(x) = \frac{1}{3} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2} - \frac{1}{n}\right)\cos n\pi x$$

(b) $f(x) = x^3;\ -\pi \prec x \prec \pi$

<u>Solution:</u>



$f(x)$ is an odd function, $a_0 = a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin\frac{nx}{\pi}dx$$

$u = x^3$	$v' = \sin nx$
$u' = 3x^2$	$v = -\frac{\cos nx}{n}$

$$b_n = \frac{2}{\pi}\left[\left[-\frac{x^3\cos nx}{n}\right]_0^{\pi} - \frac{3}{n}\int_0^{\pi} x^2\cos nx dx\right]$$

$u = x^2$	$v' = \cos nx$
$u' = 2x$	$v = -\frac{\sin nx}{n}$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{3}{n}\left[\left[\frac{x^2\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{2x\sin nx}{n}dx\right]\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{3}{n}\left[\frac{\pi^2\sin n\pi}{n} - \frac{2}{n}\int_0^{\pi} x\sin nx dx\right]\right]$$

$u = x$	$v' = \sin nx$
$u' = 1$	$v = \frac{\sin nx}{n}$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{3}{n^2}\pi^2\cos n\pi - \frac{6}{n^2}\left[\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin x dx\right]\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} - \frac{6}{n^2}\left[\frac{\pi\sin n\pi}{n} - \frac{1}{n^2}\left[-\cos nx\right]_0^{\pi}\right]\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} - \frac{6\pi\sin n\pi}{n^3} + \frac{-\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{\pi^3(-1)^n}{n} - \frac{1}{n^2}\right]$$

$$\therefore f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2} - \frac{\pi^3(-1)^n}{n} - \frac{1}{n^2}\right]\sin(nx)$$

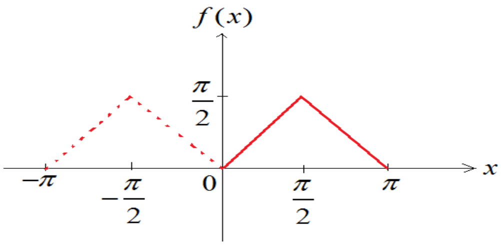
Question 6

Find the half-range Cosine and Sine expansions of the given function.

$$f(x)=\begin{cases} x & ; \quad 0 \prec x \prec \dfrac{\pi}{2} \\ \pi - x & ; \quad \dfrac{\pi}{2} \prec x \prec \pi \end{cases}$$

Solution:

Half-range cosine expansion:



$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} (x)dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{2}} + \left[\pi x - \frac{x^2}{2}\right]_{\frac{x}{2}}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left[\left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{4}\right) = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos nx dx + \int_{\frac{\pi}{2}}^{\pi} (\pi - x)\cos nx dx\right]$$

$$= \frac{2}{\pi}\left[\underbrace{\int_0^{\frac{\pi}{2}} x\cos nx dx}_{case1} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \pi\cos nx dx}_{case2} - \underbrace{\int_{\frac{\pi}{2}}^{\pi} x\cos nx dx}_{case3}\right]$$

Case 1:

$$\int_0^{\frac{\pi}{2}} x\cos nx dx = \left[\frac{x\sin(nx)}{n}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(nx)}{n}dx$$

$$= \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left[\cos(nx)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

Case 2:

$$\int_{\frac{\pi}{2}}^{\pi} \pi\cos nx dx = \frac{\pi}{n}\left[\sin n\pi\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{n}\left(\sin n\pi - \sin\frac{n\pi}{2}\right)$$

Case 3:

$$\int_{\frac{\pi}{2}}^{\pi} x\cos nx dx = \left[\frac{x\sin n\pi}{n}\right]_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi}\left(\sin n\pi\right)dx$$

$$= \frac{\pi}{n}\sin n\pi - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\left[\cos nx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{n}\sin n\pi - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\cos\frac{n\pi}{2}$$

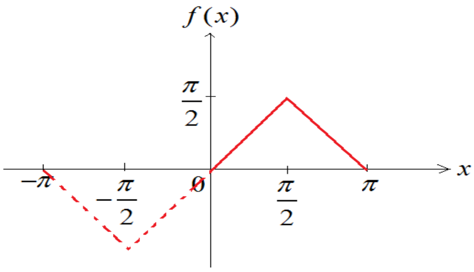
$$a_n = \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{\pi}{n}\sin\frac{n\pi}{2} + \frac{\pi}{2n}\sin\frac{n\pi}{2} - \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{2\cos\cos\frac{n\pi}{2} - (-)^n - 1}{n^2}\right)$$

$$\therefore\ f(x) = \frac{\pi}{4} + \frac{3}{4}\sum_{n=1}^{\infty}\left(\frac{2\cos\cos\frac{n\pi}{2} - (-)^n - 1}{n^2}\right)\cos nx$$

<u>Half-Range Sine Expansion:</u>



$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin\frac{n\pi x}{\pi}dx$$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin(nx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\underbrace{\int_0^{\frac{\pi}{2}} x\sin(nx)dx}_{case1} + \underbrace{\int_{\frac{\pi}{2}}^{\pi} \pi\sin(nx)\,dx}_{case2} - \underbrace{\int_{\frac{\pi}{2}}^{\pi} x\sin(nx)\,dx}_{case3}\right]$$

Case 1:

$$\int_0^{\frac{\pi}{2}} x\sin(nx)dx = \left[\frac{x\cos(n\pi)}{n}\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{\cos(n\pi)}{n}dx$$

$$= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\left[\sin(nx)\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}$$

Case 2:

$$\int_{\frac{\pi}{2}}^{0} \pi\sin(nx)dx = \frac{\pi}{n}\left[x\cos(n\pi)\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{n}\left(-\cos n\pi + \cos\frac{n\pi}{2}\right)$$

Case 3:

$$\int_{\frac{\pi}{2}}^{\pi} x\sin\left(nx\right)dx = \left[\frac{-x\cos nx}{n}\right]_{\frac{\pi}{2}}^{\pi} + \int_{\frac{\pi}{2}}^{\pi} \frac{\cos nx}{n}dx$$

$$= -\frac{\pi}{n}\cos n\pi + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \left[\sin\frac{nx}{n^2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{\pi}{n}\cos n\pi + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \sin\frac{n\pi}{n^2} - \frac{\sin\frac{n\pi}{2}}{n^2}$$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\cos n\pi + \frac{\pi}{n}\cos n\pi - \frac{2}{2\pi}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left(\frac{2}{n^2}\sin\frac{n\pi}{2}\right)$$

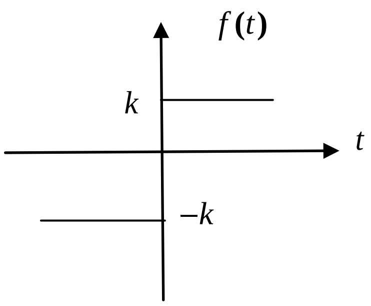
$$\therefore\ f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin(nx)$$

Question 7

A mass-spring system, in which a piston is attached to the end of the spring and drives the system.The driving force acts on the piston is represented by $f(t)$ .Find the Fourier series of $f(t)$.

$$f(t) = \begin{cases} -k\ , & -\pi \prec t \prec 0 \\ k\ , & 0 \prec t \prec \pi \end{cases}$$

$$f(t) = \begin{cases} -k & ,-\pi < t < 0 \\ k & ,0 < t < \pi \end{cases}$$



Fourier sines series

$$b_n = \frac{2}{L}\int_0^L f(x)\,\mathbf{sin}\,\frac{n\text{p}t}{L}\,dx$$

$$= \frac{2}{\text{p}}\int_0^{\text{p}} k\,\mathbf{sin}\,nt\,dx$$

$$= \frac{2}{\text{p}}\left[\frac{-k\,\mathbf{cos}\,nt}{n}\right]_0^{\text{p}}$$

$$= \frac{2}{\text{p}}\left[\frac{-k\,\mathbf{cos}\,n\text{p}+k}{n}\right]$$

$$= \frac{2k}{\text{p}}\left[\frac{-(-1)^n+1}{n}\right]$$

$$\therefore f(t) = \sum_{n=1}^{\infty}\frac{2k}{\text{p}}\left[\frac{-(-1)^n+1}{n}\right]\mathbf{sin}\,nt$$

Question 8

Find the Fourier series of the given input signal.

$$u(t) = \begin{cases} 0 & , \quad -\frac{T}{2} \prec t \prec 0 \\ E\sin\omega t & , \quad 0 \prec t \prec \frac{T}{2} \end{cases}$$

Where E is the constant and $T = \frac{2\pi}{\omega}$

$$u(t) = \begin{cases} 0 & ,\frac{-T}{2} < t < 0 \\ E\sin\omega t & ,0 < t < \frac{T}{2} \end{cases}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} E\,\mathbf{sin}\,\text{w}t\,dx$$

$$= \frac{E}{T\text{w}}\left[-\mathbf{cos}\,\text{w}t\right]_0^{\frac{T}{2}} \qquad T = \frac{2\pi}{\omega}$$

$$= \frac{E}{\text{p}}$$

$$a_n = \frac{1}{L}\int_0^L f(x)\,\mathbf{cos}\,\frac{n\text{p}x}{L}\,dt$$

$$= \frac{E}{T}\int_0^{\frac{T}{2}} \mathbf{sin}\,\text{w}t\;\mathbf{cos}\,n\text{w}t\;dt$$

$\sin\omega t$	$\cos n\omega t$
$-\omega\cos\omega t$	$\dfrac{\sin n\omega t}{n\omega}$
$-\omega^2\sin\omega t$	$\dfrac{-\cos n\omega t}{(n\omega)^2}$

$$\int_0^{\frac{T}{2}} \sin\omega t\,\cos n\omega t\;dt = \sin\omega t\left(\frac{\sin n\omega t}{n\omega}\right) - \omega\cos\omega t\left(\frac{\cos n\omega t}{(n\omega)^2}\right) +$$

$$\frac{\omega^2}{(n\omega)^2}\int_0^{\frac{T}{2}} \sin\omega t\,\cos n\omega t\;dt$$

$$\left(-\frac{\omega^2}{(n\omega)^2}+1\right)\int_0^{\frac{T}{2}} \sin\omega t\,\cos n\omega t\;dt = \sin\omega t\left(\frac{\sin n\omega t}{n\omega}\right) - \omega\cos\omega t\left(\frac{\cos n\omega t}{(n\omega)^2}\right)$$

$$\int_0^{\frac{T}{2}} \sin\omega t\,\cos n\omega t\;dt = (\sin\omega t\left(\frac{\sin n\omega t}{n\omega}\right) - \omega\cos\omega t\left(\frac{\cos n\omega t}{(n\omega)^2}\right))_0^{\frac{T}{2}} \;/ -\left(\frac{\omega^2}{(n\omega)^2}+1\right)$$

$$= \left[\frac{1}{n^2\omega}\cos\pi\cos n\pi - \frac{\omega}{(n\omega)^2}\right] / \left(\frac{n^2-1}{(n)^2}\right)$$

$$= \left(\frac{(-1)^n+1}{n^2-1}\right)$$

$$a_n = \frac{E}{T}\left(\frac{(-1)^n+1}{n^2-1}\right)$$

$$= \frac{E}{2\text{p}}\left(\frac{(-1)^n+1}{n^2-1}\right)$$

$$b_n = \frac{1}{L}\int_0^L f(x)\,\mathbf{sin}\,\frac{np x}{L}\,dt$$

$$= \frac{E}{T}\int_0^{\frac{T}{2}} \mathbf{sin}\,wt\;\mathbf{sin}\,nwt\;dt$$

$$= \frac{E}{T}\int_0^{\frac{T}{2}} \frac{1}{2} - \frac{\mathbf{cos}\,2wt}{2}\quad dx$$

$$= \frac{E}{T}\left[\frac{1}{2}t - \frac{\mathbf{sin}\,2wt}{4w}\right]_0^{\frac{T}{2}}$$

$$= \frac{E}{\frac{2p}{w}}\left[\frac{p}{2w}\right] = \frac{1}{4}E$$

$$u(t) = \frac{E}{p} - \frac{E}{2p}\sum_{n=1}^{\infty}\left(\frac{(-1)^n + 1}{n^2 - 1}\right)\mathbf{cos}\,nwt + \frac{1}{4}E\,\mathbf{sin}\,wt$$

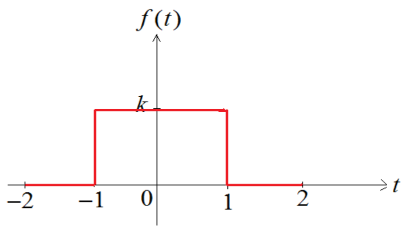
Question 9

Find the Fourier series of the given function.

$$f(t) = \begin{cases} 0\ , & -2 \prec t \prec -1 \\ k\ , & -1 \prec t \prec 1 \\ 0\ , & 1 \prec t \prec 2 \end{cases}$$

Period, $T = 4$

Solution:



Even function => Fourier Cosine Series, $b_n = 0$

$$
\begin{aligned}
a_0 &= \frac{2}{L}\int_0^L f(x)\,dx \\
&= \frac{2}{2}\int_0^1 k\,dx \\
&= \left[kt\right]_0^1 \\
&= k
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(x)\,\mathbf{cos}\,\frac{n\pi x}{L}\,dt \\
&= \frac{2}{2}\int_0^1 k\,\mathbf{cos}\,\frac{n\pi t}{2}\,dt \\
&= k\left[\mathbf{sin}\,\frac{n\pi t}{2}\times\frac{2}{n\pi}\right]_0^1 \\
&= \frac{2k}{n\pi}\,\mathbf{sin}\,\frac{n\pi}{2}
\end{aligned}
$$

$$
f(t) = \frac{k}{2} + \sum_{n=1}^{\infty}\frac{2k}{n\pi}\sin\frac{n\pi}{2}\,\cos\frac{n\pi t}{2}
$$