CHAPTER THREE

MATRICES

Introduction

In this chapter, we will discuss some operations that can be performed on matrices. These include matrix addition and multiplication. Important matrix operations such as transposition and inversion will also be studied. It will also apparent in this chapter that the matrix is different from real numbers, as some properties of real numbers are not satisfied by a matrix even though the entries of a matrix are real numbers.

Objectives

After completing these tutorials, students should be able to:

- Simplify matrix expressions.
- Find transposition of the given matrices.
- Find minor and cofactor of the given matrices.
- Find the determinant of the given matrix by using cofactor expansion.
- Find the inverse matrix by using adjoint method.
- Express the given linear equations system in the matrix form and solve that system by using inverse matrix.
- Express the given linear equations system in the matrix form and solve that system by using Cramer’s rule.

Question 1

Find the values of a, b, c *and* d if:

(a) $\begin{pmatrix} 2-a & 3 & 4 \\ 4 & 2b & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3-c & 4 \\ 2-4d & -8 & 2 \end{pmatrix}$

Solution:

$2-a=6$ $\quad 2b=-8$ $\quad 3-c=3$ $\quad 2-4d=4$

$-a=4$ $\quad b=-4$ $\quad -c=0$ $\quad -4d=2$

$a=-4$ $\quad c=0$ $\quad d=-\frac{1}{2}$

$\therefore \; a=3, \quad b=-4, \quad c=0, \quad d=-\frac{1}{2}$

(b) $2\begin{pmatrix} 2 & a \\ -3d & 4+a \end{pmatrix} = 3\begin{pmatrix} c & 2 \\ 1 & 2-b \end{pmatrix}$

Solution:

$$\begin{pmatrix} 4 & 2a \\ -6d & 8+2a \end{pmatrix} = \begin{pmatrix} 3c & 6 \\ 3 & 6-2b \end{pmatrix}$$

$2a=6$ $\quad 8+2b=6-3b$ $\quad 3c=4$ $\quad -6d=3$

$a=3$ $\quad b=-\frac{2}{5}$ $\quad c=\frac{4}{3}$ $\quad d=-\frac{1}{2}$

$\therefore \; a=3, \quad b=-\frac{2}{5}, \quad c=0, \quad d=-\frac{1}{2}$

Question 2

Given that $A=\left[\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}\right]$, $B=\left[\begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix}\right]$, $C=\left[\begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix}\right]$. Simplify each of the following:

(a) $3A-6B+9C$

Solution:

$$=3\left[\begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{pmatrix}\right]-6\left[\begin{pmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{pmatrix}\right]+9\left[\begin{pmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{pmatrix}\right]$$

$$=\begin{bmatrix} 6 & 6 & 6 \\ 6 & 3 & -9 \\ 3 & 0 & 12 \end{bmatrix}-\begin{bmatrix} 18 & 18 & 18 \\ 18 & 0 & 30 \\ 36 & 54 & 6 \end{bmatrix}+\begin{bmatrix} 36 & 36 & 36 \\ 40 & -9 & 0 \\ 63 & 72 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 6-18+26 & 6-18+36 & 6-18+36 \\ 6-18+45 & 3-0-9 & -9-30+0 \\ 3-36+63 & 0-54+72 & 12+6-9 \end{bmatrix} \qquad = \begin{bmatrix} 24 & 24 & 24 \\ 33 & -6 & -39 \\ 30 & 18 & 9 \end{bmatrix}$$

(b) $7A - 2(B - C)$

Solution:

$$= 7\begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & -3 \\ 1 & 0 & 4 \end{bmatrix} - 2\left(\begin{bmatrix} 3 & 3 & 3 \\ 3 & 0 & 5 \\ 6 & 9 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 4 & 4 \\ 5 & -1 & 0 \\ 7 & 8 & -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 14 & 14 & 14 \\ 14 & 7 & -21 \\ 7 & 0 & 28 \end{bmatrix} - 2\begin{bmatrix} -1 & -1 & -1 \\ -2 & 1 & 5 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 14 & 14 \\ 14 & 7 & -21 \\ 7 & 0 & 28 \end{bmatrix} - \begin{bmatrix} -2 & -2 & -2 \\ -4 & 2 & 10 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 16 & 16 \\ 18 & 5 & -31 \\ 9 & -2 & 28 \end{bmatrix}$$

Question 3

Given that $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 & 0 \\ 3 & -6 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ -5 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix}$.

Evaluate the following:

(a) $BC + 2A$

Solution:

$$= \begin{bmatrix} 1 & -3 & 0 \\ 3 & -6 & -2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+(-3)(4)+(0)(-5) & (1)(-5)+(-3)(-3)+(0)(2) \\ 3(1)+(-6)(4)+(-2)(-5) & (3)(-5)+(-6)(-3)+(-2)(2) \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 4 \\ -11 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix} \qquad = \begin{bmatrix} -9 & 2 \\ -11 & 3 \end{bmatrix}$$

(b) $D^2 - CB$

Solution:

$$= \left(\begin{bmatrix} 3 & 4 & 0 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} \right)^2 - \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 3 & -6 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 0 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} (1)+(-15) & -3+30 & 0+10 \\ 4-9 & -12+18 & 0+6 \\ -5+6 & 15-12 & 0-4 \end{bmatrix}$$

$$= \begin{bmatrix} 9+8+0 & 12+16+0 & 0+12+0 \\ 6+8+3 & 8+16+9 & 0+12+6 \\ 3+6+2 & 4+12+16 & 0+9+4 \end{bmatrix} - \begin{bmatrix} -14 & 27 & 10 \\ -5 & 6 & 6 \\ 1 & 3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 1 & 2 \\ 22 & 27 & 12 \\ 10 & 19 & 17 \end{bmatrix}$$

(c) $A - 3(BC + A)$

Solution:

$$= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} - 3\left(\begin{bmatrix} 1 & -3 & 0 \\ 3 & -6 & -2 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} - 3\left(\begin{bmatrix} -11 & 4 \\ -11 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} -30 & 9 \\ -33 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 31 & -10 \\ 33 & -1 \end{bmatrix}$$

Question 4

Given that $A = \begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.

Find:

(a) $A^T B$

Solution:

$$= \begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}^T \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ 6 & 31 \end{bmatrix}$$

(b) $B^T A$

Solution:

$$= \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}^T \begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ -2 & 31 \end{bmatrix}$$

(c) $(BC)^T$

Solution:

$$= \left(\begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} 15 & 18 \\ 4 & 8 \end{bmatrix}^T$$

$$= \begin{bmatrix} 15 & 4 \\ 18 & 8 \end{bmatrix}$$

(d) $(A+B)^T$

Solution:

$$=\left(\begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}+\begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}\right)^T$$

$$=\begin{bmatrix} 4 & 8 \\ -3 & 8 \end{bmatrix}^T$$

$$=\begin{bmatrix} 4 & -3 \\ 8 & 8 \end{bmatrix}$$

Question 5

If $A=\begin{bmatrix} 2 & 1 & 3 \\ 4 & 8 & 6 \\ 0 & 7 & 5 \end{bmatrix}$, find the minor m_{21}, m_{32}, m_{23}.

Solution:

$$m_{21}=\begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix}=-16$$

$$m_{32}=\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}=0$$

$$m_{23}=\begin{vmatrix} 2 & 1 \\ 0 & 7 \end{vmatrix}=14$$

Question 6

Write the cofactor matrix of the followings:

(a) $A = \begin{bmatrix} 2 & -4 & -2 \\ -2 & 5 & 4 \\ 4 & 1 & -3 \end{bmatrix}$

Solution:

$C_{11} = (-1)^{1+1}\begin{vmatrix} 5 & 4 \\ 1 & -3 \end{vmatrix} = 19$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 4 \\ 4 & -3 \end{vmatrix} = 10$ $\quad C_{13} = (-1)^{1+3}\begin{vmatrix} -2 & 5 \\ 4 & 1 \end{vmatrix} = 22$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -4 & -2 \\ 1 & -3 \end{vmatrix} = -14$ $\quad C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & -2 \\ 4 & -3 \end{vmatrix} = 2$ $\quad C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -4 \\ 4 & 1 \end{vmatrix} = -18$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -4 & -2 \\ 5 & 4 \end{vmatrix} = -6$ $\quad C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = -4$ $\quad C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -4 \\ -2 & 5 \end{vmatrix} = 2$

$$\therefore Cofactor\ \ matrix\, A = \begin{pmatrix} -19 & 10 & -22 \\ -14 & 2 & -18 \\ -6 & -4 & 2 \end{pmatrix}$$

(b) $B = \begin{bmatrix} 4 & 10 & -2 \\ 8 & 3 & -5 \\ 6 & -3 & 3 \end{bmatrix}$

Solution:

$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -5 \\ -3 & 3 \end{vmatrix} = -6$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} 8 & -5 \\ 6 & 3 \end{vmatrix} = -54$ $\quad C_{13} = (-1)^{1+3}\begin{vmatrix} 8 & 3 \\ 6 & -3 \end{vmatrix} = -42$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 10 & -2 \\ -3 & 3 \end{vmatrix} = -24$ $\quad C_{22} = (-1)^{2+2}\begin{vmatrix} 4 & -2 \\ 6 & 3 \end{vmatrix} = 24$ $\quad C_{23} = (-1)^{2+3}\begin{vmatrix} 4 & 10 \\ 6 & -3 \end{vmatrix} = 72$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 10 & -2 \\ 3 & -5 \end{vmatrix} = -44$ $\quad C_{32} = (-1)^{3+2}\begin{vmatrix} 4 & -2 \\ 8 & -5 \end{vmatrix} = 4$ $\quad C_{33} = (-1)^{3+3}\begin{vmatrix} 4 & 10 \\ 8 & 3 \end{vmatrix} = -68$

$$\therefore Cofactor\ \ matrix\, B = \begin{pmatrix} -6 & -54 & -42 \\ -24 & 24 & 72 \\ -44 & 4 & -68 \end{pmatrix}$$

Question 7

Given the matrix $A=\begin{bmatrix} 2 & -4 & -2 \\ -2 & 5 & 4 \\ 4 & 1 & -3 \end{bmatrix}$, find the determinant using cofactor expansion of the :

(a) Second row

Solution:

$$|A| = (-2)(-1)^{2+1}\begin{vmatrix} -4 & -2 \\ 1 & -3 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 2 & -2 \\ 4 & -3 \end{vmatrix} + 4(-1)^{2+3}\begin{vmatrix} 2 & -4 \\ 4 & 1 \end{vmatrix}$$

$$= (-2)(-14) + (5)(2) + (4)(-18)$$

$$= 28 + 10 - 72$$

$$= -34$$

(b) Third column

Solution:

$$|A| = (-2)(-1)^{1+3}\begin{vmatrix} -2 & 5 \\ 4 & 1 \end{vmatrix} + (4)(-1)^{2+3}\begin{vmatrix} 2 & -4 \\ 4 & 1 \end{vmatrix} + (-3)(-1)^{3+3}\begin{vmatrix} 2 & -4 \\ -2 & 5 \end{vmatrix}$$

= (-2)(-22)+(4)(-18)+(-3)(2)
= 44-72-6
= -34

(c) Second column

Solution:

$$|A| = (-4)(-1)^{1+2}\begin{vmatrix} -2 & -4 \\ 4 & -3 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 2 & -2 \\ 4 & -3 \end{vmatrix} + (1)(-1)^{3+2}\begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix}$$

$$= (-4)(10) + (5)(2) + (1)(-4)$$

$$= -40 + 10 - 4$$

$$= -34$$

Question 8

Find the inverse matrix of the following using adjoint method:

(a)
$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & -5 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & -5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$|A| = 2(-2-5) - 1(8+5) + 0$$
$$= -27$$

Cofactor matrix of A is

$$= \begin{bmatrix} \begin{vmatrix} -1 & -5 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -5 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -1 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & -5 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -13 & -3 \\ -2 & 4 & 3 \\ -5 & 10 & -6 \end{bmatrix}$$

$$\text{Adj (A)} = \begin{bmatrix} -7 & -2 & -5 \\ -13 & 4 & 10 \\ -3 & 3 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-27}\begin{bmatrix} -7 & -2 & -5 \\ -13 & 4 & 10 \\ -3 & 3 & -6 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 7 & 2 & 5 \\ 13 & -4 & -10 \\ 3 & -3 & 6 \end{bmatrix}$$

(b) $$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 1\\ 1 & -1 & 2\end{bmatrix}$$

Solution:

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 1\\ 1 & -1 & 2\end{bmatrix}$$

$$\begin{aligned}|A| &= 2(4+1)-3(2-1)+1(-1-2)\\ &=10-3-3\\ &=4\end{aligned}$$

Cofactor matrix of A is

$$=\begin{bmatrix}\begin{vmatrix}2 & 1\\ -1 & 2\end{vmatrix} & -\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix} & \begin{vmatrix}1 & 2\\ 1 & -1\end{vmatrix}\\ -\begin{vmatrix}3 & 1\\ -1 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix}\\ \begin{vmatrix}3 & 1\\ 2 & 1\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & -3\\ -7 & 3 & 5\\ 1 & -1 & 1\end{bmatrix}$$

$$Adj\ (A)=\begin{bmatrix}5 & -7 & 1\\ -1 & 3 & -1\\ -3 & 5 & 1\end{bmatrix}$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}5 & -7 & 1\\ -1 & 3 & -1\\ -3 & 5 & 1\end{bmatrix}$$

Question 9

Given that matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 3 & 7 \end{bmatrix}; \ B = \begin{bmatrix} 6 & 4 & -2 \\ 3 & -5 & 1 \\ -3 & 1 & 1 \end{bmatrix}$. Prove that AB = kI, where I is an identity matrix and k is a constant.

<u>Solution</u>:

$$AB = kI$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 3 & 7 \end{bmatrix}\begin{bmatrix} 6 & 4 & -2 \\ 3 & -5 & 1 \\ -3 & 1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore k = 6 \ \textit{and I is an identity matrix}$$

Question 10

If $P=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}$; $Q=\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, evaluate PQ and find the matrix P^{-1}. Express the following system in the matrix form and solve for x, y and z.

$2x-y+z=3$

$x+z=1$

$3x-y+4z=0$

Solution:

$$PQ=\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 & -1 \\ -1 & -5 & -1 \\ -1 & -1 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|P|=-1(-4+1)+0-1(-2+3)=2$$

Cofactor matrix of P is

$$=\begin{bmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & -1 & -1 \\ -3 & 5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\text{Adj (P)} =\begin{bmatrix} 1 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$P^{-1}=\frac{1}{2}\begin{bmatrix} 1 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$2x-y+z=3$

$x+z=1$

$3x-y+4z=0$

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P^{-1}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 3 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\therefore x = 3,\ y = 1,\ z = -2$$

Question 11

Solve the given systems of linear equation using:

(a) Inverse matrix

Solution:

$$x - 2y + z = 5$$
$$-2x + 3y + 2z = 1$$
$$x - 3y + 2z = 2$$

$$\underbrace{\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}}_{B}$$

$$|A| = -21$$

Cofactor of matrix A is:

$$= \begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 6 & -9 \\ 7 & 1 & -5 \\ -7 & -4 & -1 \end{bmatrix}$$

$$\text{Adj (A)} = \begin{bmatrix} 0 & 7 & -7 \\ 6 & 1 & -4 \\ -9 & -5 & -1 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{21}\begin{bmatrix} 0 & -7 & 7 \\ -6 & -1 & 4 \\ 9 & 5 & 1 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 0 & -7 & 7 \\ -6 & -1 & 4 \\ 9 & 5 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 7 \\ -23 \\ 52 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -1\,2/21 \\ 2\,10/21 \end{bmatrix}$$

$$x = 1/3, \quad y = -1\,2/21, \quad z = 2\,10/21$$

(b) Cramer's rule

Solution:

$$x - 3y + 2z = 1$$

$$5x + 2y + 7z = -2$$

$$x + 2y + z = 2$$

$$\underbrace{\begin{bmatrix} 1 & 3 & 2 \\ 5 & 2 & 7 \\ 1 & 2 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}_{B}$$

$$|A| = \begin{vmatrix} 1 & 3 & 2 \\ 5 & 2 & 7 \\ 1 & 2 & 1 \end{vmatrix} = 10$$

$$X = \frac{1}{10}\begin{vmatrix} 1 & 3 & 2 \\ -2 & 2 & 7 \\ 2 & 2 & 1 \end{vmatrix} = 2$$

$$Y = \frac{1}{10}\begin{vmatrix} 1 & 1 & 2 \\ 5 & -2 & 7 \\ 1 & 2 & 1 \end{vmatrix} = 1$$

$$Z = \frac{1}{10}\begin{vmatrix} 1 & 3 & 1 \\ 5 & 2 & -2 \\ 1 & 2 & 2 \end{vmatrix} = -2$$

$$x = 2, \quad y = 1, \quad z = -2$$