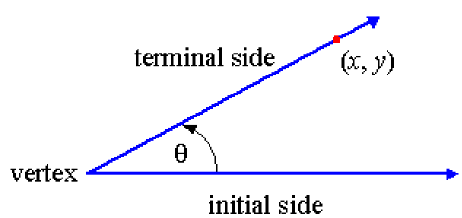
CHAPTER THREE
TRIGONOMETRY

3.0 Introduction

Trigonometry is a branch of mathematics that studies triangles, particularly right triangles. Trigonometry deals with relationships between the sides and the angles of triangles and with the trigonometric functions, which describe those relationships, as well as describing angles in general and the motion of waves such as sound and light waves.
It has applications in both pure mathematics and in applied mathematics, where it is essential in many branches of science and technology.

3.1 Angles

An **angle** is the amount of rotation between two line segments. We name these 2 line segments (or **rays**) the **initial side** and **terminal side.**

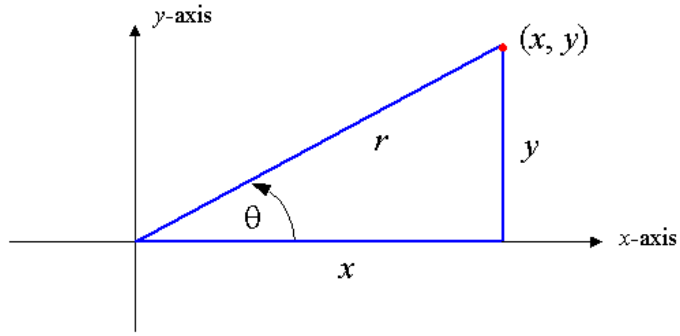


In the diagram, we say that the terminal side passes through the point (x, y).

If the rotation is anti-clockwise, the angle is positive. Clockwise rotation gives a negative angle.

Angles can be measured in degrees or radians (also **gradians**, but these are not common).

3.2 Standard Position of an Angle



An angle is in **standard position** if the initial side is the positive x-axis and the vertex is at the origin.

3.3 Degrees, Minutes and Seconds

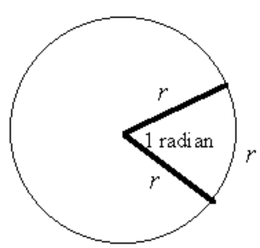
The Babylonians (who lived in modern day Iraq from 5000 BC to 500 BC) used a base 60 system of numbers. From them we get our divisions of time and also angles.

A **degree** is divided into 60 minutes (') and a minute is divided into 60 seconds ("). We can write this form as: DMS or ° ' ".

3.4 Radians

In science and engineering, **radians** are much more convenient (and common) than degrees. A radian is defined as the angle between 2 radii of a circle where the arc between them has length of one radius.

A **radian** is the angle subtended by an arc of length r (the radius):



One radian is about 57.3°.

Since the circumference of a circle is $\boldsymbol{2\pi r}$, it follows that

2π radians = 360°.

Also, **π radians = 180°**.

3.5 Converting Degrees to Radians

Because the circumference of a circle is given by $C = 2\pi r$ and one revolution of a circle is 360°, it follows that 2π radians = 360°

This gives us the important result:

π radians = 180°

From this we can convert:

radians → degrees

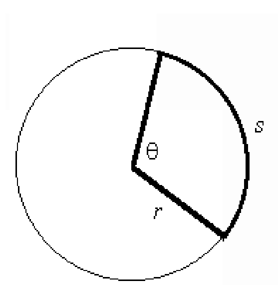
$$\boxed{y \text{ rad} \xrightarrow{y\left(\frac{180^\circ}{\pi}\right)} \text{degree } (^\circ)}$$

degrees → radians.

$$\boxed{x^\circ \xrightarrow{x^\circ\left(\frac{\pi}{180^\circ}\right)} \text{rad}}$$

3.6 Applications of the Use of Radian Measure

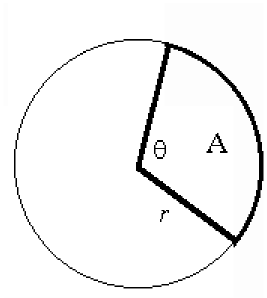
3.6.1 Arc Length



The length, s, of an arc of a circle radius r subtended by θ (in *radians*) is given by:

$s = r\theta$

3.6.2 Area of a Sector



The area of a sector with central angle θ (in radians) is given by:

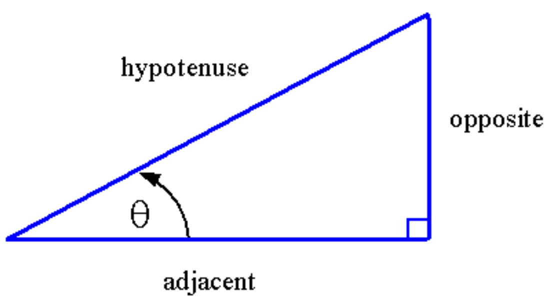
$$\text{Area} = \frac{\theta r^2}{2}$$

3.6.3 Angular Velocity

The time rate of change of angle θ by a rotating body is the **angular velocity**, written ω (omega). It is measured in radians/second.

If *v* is the **linear velocity** (in m/s) and *r* is the radius of the circle, then $\boldsymbol{v = r\omega}$.

3.7 Sine, Cosine, Tangent and the Reciprocal Ratios



For the angle θ in a right-angled triangle as shown, we name the sides as:

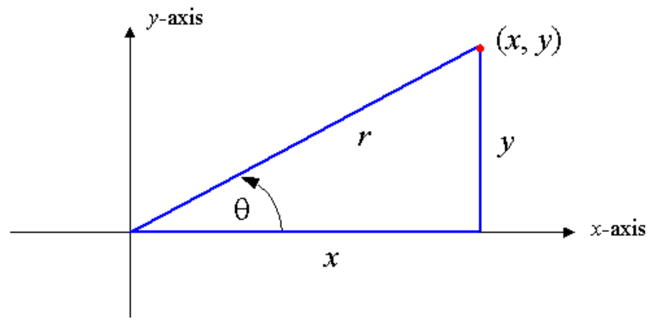
- hypotenuse (opposite the right angle)
- adjacent ("next to" θ)
- opposite

We **define** the six trigonometrical ratios as:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$



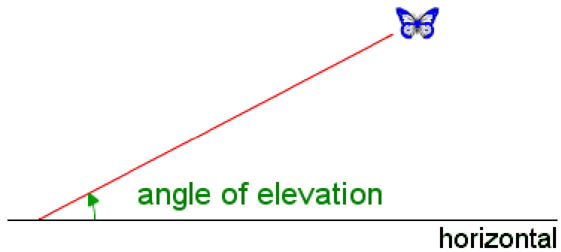
For an angle in **standard position**, we define the ratios in terms of *x*, *y* and *r*:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

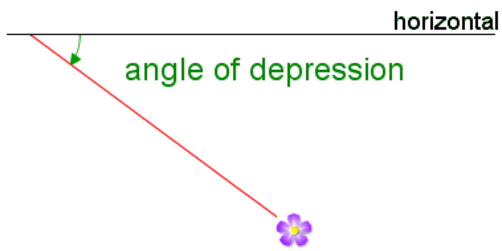
$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

3.8 Angle of Elevation and Depression

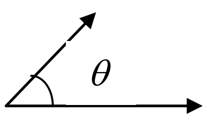
In surveying, the **angle of elevation** is the angle from the horizontal looking **up** to some object:



The **angle of depression** is the angle from the horizontal looking **down** to some object:

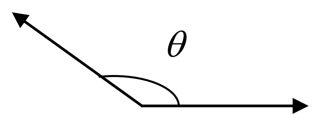


3.9 Types of Angle



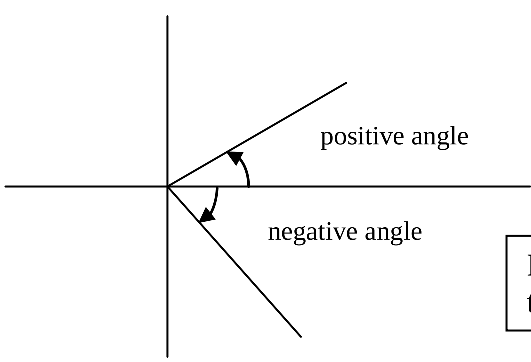
Acute angle

Right angle



Obtuse angle

3.9.1 Positive and Negative Angles



Positive angle is measured in the counter-clockwise direction

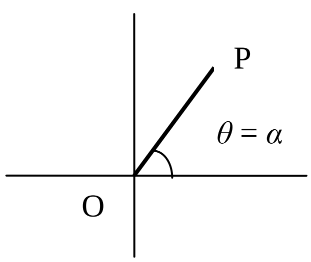
Negative angle is measured in the clockwise direction

3.9.2 Four Quadrants- Signs of the Trigonometric Functions

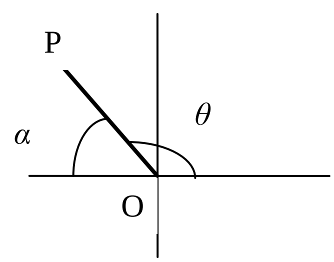
<u>**Quadrant II**</u> ($90^0 < \theta < 180^0$) $\left(\frac{\pi}{2} < \theta < \pi\right)$ **sin** has **+ve** value	<u>**Quadrant I**</u> ($0^0 < \theta < 90^0$) $\left(0 < \theta < \frac{\pi}{2}\right)$ **sin, cos, tan** have **+ve** values
<u>**Quadrant III**</u> ($180^0 < \theta < 270^0$) $\left(\pi < \theta < \frac{3}{2}\pi\right)$ **tan** has **+ve** value	<u>**Quadrant IV**</u> ($270^0 < \theta < 360^0$) $\left(\frac{3}{2}\pi < \theta < 2\pi\right)$ **cos +ve**

3.9.3 Reference Angle

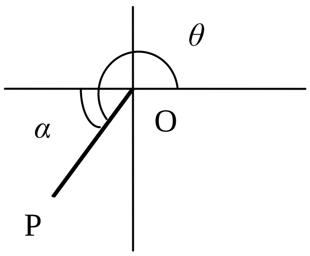
Let θ denote a nonacute angle that lies in a quadrant. The acute angle formed by the line OP to the x-axis is called the reference angle for θ. We denoted **reference angle** for θ as α.



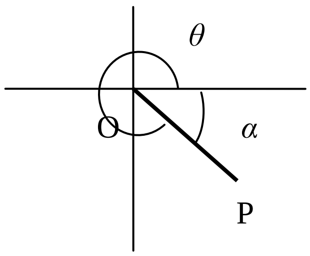
OP in quadrant I: $\alpha = \theta$



OP in quadrant II: $\theta = 180^o - \alpha$



OP in quadrant III: $\theta = 180^{o} + \alpha$



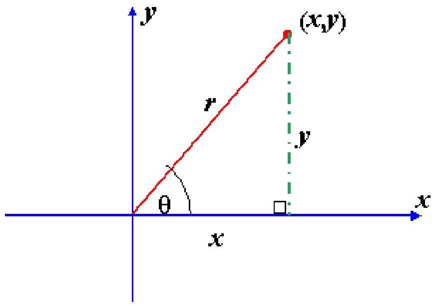
OP in quadrant IV: $\theta = 360^{o} - \alpha$

3.10 Fundamental Trigonometric Identities

Recall the definitions:

$\cot\theta = \dfrac{1}{\tan\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\csc\theta = \dfrac{1}{\sin\theta}$

Now, consider the following:



From the diagram, we can conclude the following:

Since $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$ then:

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}$$

Now, also $\tan\theta = \frac{y}{x}$ so we can conclude that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Also, from the diagram, we can use Pythagoras' Theorem and obtain:

$$y^2 + x^2 = r^2$$

Dividing through by r^2 gives us:

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1$$

implies that

$$\sin^2\theta + \cos^2\theta = 1$$

Dividing $\sin^2\theta + \cos^2\theta = 1$ through by $\cos^2\theta$ gives us:

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Dividing $\sin^2\theta + \cos^2\theta = 1$ through by $\sin^2\theta$ gives us:

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

So

$$1 + \cot^2\theta = \csc^2\theta$$

3.10.1 Trigonometric Identities Summary

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

3.10.2 Proving Trigonometric Identities

Suggestions...

- **Learn well** the formulas given above
- Work on the *most complex* side and simplify it so that it has the same form as the simplest side.
- Don't **assume** the identity to **prove** the identity
- Many of these come out quite easily if you express everything on the most complex side in terms of sine and cosine only.
- In most examples where you see power 2 (that is, ^{2}), it will involve using the identity $\sin^2\theta + \cos^2\theta = 1$.

Using these suggestions, you can simplify an expression using trigonometric identities.

3.11 Sum and Difference of Two Angles

We can show, using the product of the sum of 2 complex numbers that:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

and

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Also

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

3.12 Double-Angle Formulas

The double-angle formulas can be quite useful when we need to simplify complicated trigonometric expressions later.

With these formulas, it is better to remember where they come from, rather than trying to remember the actual formulas. In this way, you will understand it better and have less to clutter your memory with.

3.12.1 Sine of a Double Angle

If we take

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

and replace β with α, we get on the LHS:

$$\sin(\alpha+\beta)=\sin(\alpha+\alpha)=\sin 2\alpha$$

and on the RHS:

$$\sin\alpha\cos\alpha+\cos\alpha\sin\alpha=2\sin\alpha\cos\alpha$$

This gives us the important result:

$$\sin 2\alpha=2\sin\alpha\cos\alpha$$

3.12.2 Cosine of a Double Angle

Similarly, we can derive:

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$$

By using the result $\sin^2\alpha+\cos^2\alpha=1$, we can obtain:

$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$= 1 - 2\sin^2\alpha$$

3.13 Half-Angle Formulas

Using the identity

$$\cos 2\theta = 1 - 2\sin^2\theta,$$

if we let $\theta = \frac{\alpha}{2}$ and then solve for $\sin\left(\frac{\alpha}{2}\right)$,

we get the following half-angle identity:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

The sign of $\sin\left(\frac{\alpha}{2}\right)$ depends on the quadrant in which α/2 lies.

With the same substitution of $\theta = \frac{\alpha}{2}$ in the identity

$$\cos 2\theta = 2\cos^2\theta - 1$$

we obtain:

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

The sign of $\cos\left(\frac{\alpha}{2}\right)$ depends on the quadrant in which α/2 lies.

3.14 Product-to-Sum Identities

We simply add the sum and difference identities for sine:

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \cdots\cdots(1)$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y \cdots\cdots(2)$$

(1)+(2):

$$\sin(x+y)+\sin(x-y)=2\sin x\cos y$$
$$\sin x\cos y=\frac{1}{2}\left(\sin(x+y)+\sin(x-y)\right)$$

Similarly, by adding/subtracting the sum and difference identities, we can obtain three other product-to-sum identities.

3.14.1 Product to Sum Formulas

$$\sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$$
$$\cos x\sin y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$$
$$\cos x\cos y=\frac{1}{2}[\cos(x+y)+\cos(x-y)]$$
$$\sin x\sin y=\frac{1}{2}[\cos(x-y)-\cos(x+y)]$$

3.15 Sum-to-Product Identities

The product-to-sum identities can be transformed into sum-to-product identities. Let us consider

$$\sin x\cos y=\frac{1}{2}[\sin(x+y)+\sin(x-y)]$$

Let $A=x+y\cdots\cdots(1)$

$B=x-y\cdots\cdots(2)$

$$A+B=2x$$
$$x=\frac{A+B}{2}$$

(1)-(2):

$$A-B=2y$$
$$y=\frac{A-B}{2}$$

Substituting $x = \frac{A+B}{2}, y = \frac{A-B}{2}$ into the identity, we obtain

$$\sin\frac{A+B}{2}\cos\frac{A-B}{2} = \frac{1}{2}[\sin A + \sin B]$$
$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

The other three identities can be obtained by using similar procedures.

3.15.1 Sum to Product Formulas

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

3.16 Solving Trigonometric Equations

Trigonometric equations can be solved using the algebraic methods and trigonometric identities and values discussed in earlier sections.

A painless way to solve these is using a graph. Where the graph cuts the *x*-axis, there are your solutions. Graphs also help you to understand why sometimes there is one answer, and sometimes many answers.

3.17 Expressing *a* sin θ ± *b* cos θ in the form *R* sin(θ ± α)

In electronics, we often get expressions involving the sum of sine and cosine terms. It is more convenient to write such expressions using one single term.

Problem:

Express ***a* sin θ ± *b* cos θ** in the form $R\sin(\theta \pm \alpha)$, where a, b, R and α are **positive** constants.

Solution:

We take the $(\theta + \alpha)$ case first to make things easy.

Let

$a \sin\theta + b \cos\theta \equiv R \sin(\theta + \alpha)$

(The symbol " ≡ " means: "is identically equal to")

Using the compound angle formula :

sin(A + B) = sin A cos B + cos A sin B,

we can expand the RHS of the line above as follows:

$R \sin(\theta + \alpha) \equiv R(\sin\theta \cos\alpha + \cos\theta \sin\alpha)$

$\equiv R \sin\theta \cos\alpha + R \cos\theta \sin\alpha$

So

$\boldsymbol{a} \sin\theta + \boldsymbol{b} \cos\theta \equiv \boldsymbol{R \cos\alpha} \sin\theta + \boldsymbol{R \sin\alpha} \cos\theta$

Equating the coefficients of $\sin\theta$ and $\cos\theta$ in this identity, we have:

For $\sin\theta$:
$a = R\cos\alpha$ ....(1)

For $\cos\theta$:
$b = R\sin\alpha$ ....(2)

Eq. (2) ÷ Eq.(1):

$$\frac{b}{a} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$$

So $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

(α is a positive acute angle and a and b are **positive**.)

Now we square each of Eq. (1) and Eq. (2) and add them.

[Eq. (1)]^2 + [Eq. (2)]^2:

$$\begin{aligned} a^2 + b^2 &= R^2 \cos^2 \alpha + R^2 \sin^2 \alpha \\ &= R^2 \left(\cos^2 \alpha + \sin^2 \alpha\right) \\ &= R^2 \end{aligned}$$

(since $\cos^2 A + \sin^2 A = 1$)

So

$$R = \sqrt{a^2 + b^2}$$

(We take only the positive root)

Thus, using the values for α and *R* above, we have

$$a \sin\theta + b \cos\theta = R \sin(\theta + \alpha)$$

Similarly,

$$a \sin\theta - b \cos\theta \equiv R \sin(\theta - \alpha)$$

3.17.1 Equations of the type $a \sin \theta \pm b \cos \theta = c$

Method of Solution:
Express the LHS in the form $R \sin(\theta \pm \alpha)$ and then solve **$R \sin(\theta \pm \alpha) = c$.**

3.18 The Inverse Trigonometric Functions

We define the inverse sine function as

$$y = \sin^{-1} x \text{ for } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

where *y* is the angle whose sine is *x*. This means that

$$x = \sin y$$

Writing the inverse sine function showing its range in another way, we have:

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

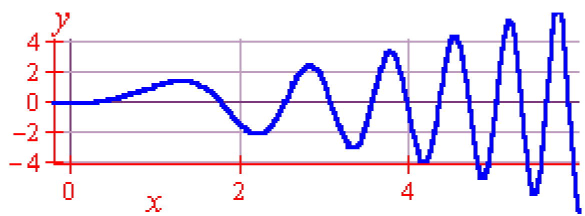
Similarly, for the other inverse trigonometric functions we have:

$$0 \le \cos^{-1} x \le \pi$$

$$-\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$$

3.19 Graphs of the Trigonometric Functions

3.19.1 Why study these trigonometric graphs?



The graphs in this section are probably the most commonly used in all areas of science and engineering. They are used for **modeling** many different natural and mechanical phenomena (populations, waves, engines, acoustics, electronics, UV intensity, growth of plants and animals, etc).

The best thing to do in this section is to **learn** the basic shapes of each graph. Then it is only a matter of considering what effect the variables are having.

3.19.2 Graphs of $y = a \sin x$ and $y = a \cos x$

Recall the shapes of the curves $\boldsymbol{y = \sin t}$ and $\boldsymbol{y = \cos t}$.

The a in both of the graph types

$\boldsymbol{y = a \sin x}$ and $\boldsymbol{y = a \cos x}$

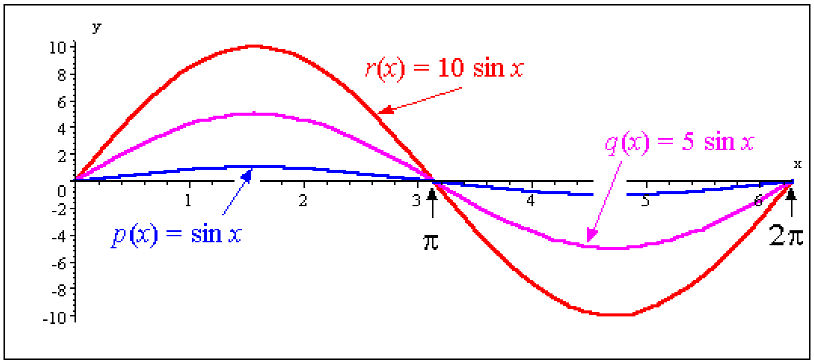
affects the **amplitude** of the graph.

In the following frame, we have graphs of

- $y = \sin x$
- $y = 5 \sin x$
- $y = 10 \sin x$

on the one set of axes.

Note that the graphs have the same PERIOD but different AMPLITUDE.

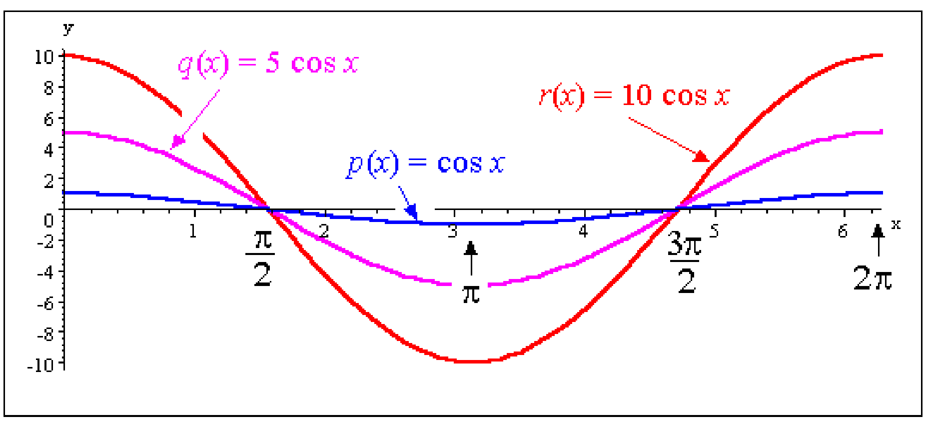


Now let's do the same for the graph of $y = \cos x$:

Here we have graphs of

- $y = \cos x$
- $y = 5 \cos x$
- $y = 10 \cos x$

on one set of axes:



Note: The **period** of each graph is the same (2π), but the **amplitude** has changed.

3.19.3 Graphs of *y* = *a* sin *bx* and *y* = *a* cos *bx*

The b in both of the graph types

- $y = a \sin bx$
- $y = a \cos bx$

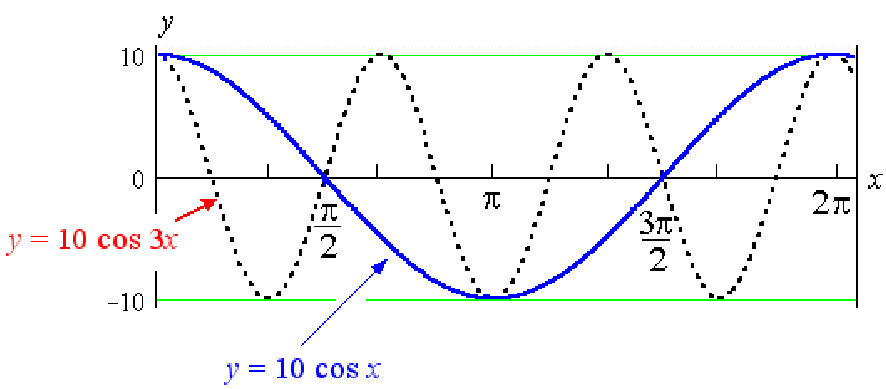
affects the **period** (or wavelength) of the graph.

The **period** is given by:

$$\text{Period} = \frac{2\pi}{b}$$

Note: As b gets larger, the period decreases.

Let's look at a graph with $y = 10 \cos x$ and $y = 10 \cos 3x$ on the same set of axes. Note that both graphs have an amplitude of 10 units, but their PERIOD is different.



Note: b tells us the number of cycles in each 2π.

For $y = 10 \cos x$, there is **one** cycle between 0 and 2π (because $b = 1$).

For $y = 10 \cos 3x$, there are **3** cycles between 0 and 2π (because $b = 3$).

3.19.4 Graphs of $y = a \sin(bx + c)$ and $y = a \cos(bx + c)$

The ***c*** (and ***b***) in the graph types

- $y = a \sin(bx + c)$
- $y = a \cos(bx + c)$

affects the **phase shift** (or **displacement**),given by:

$$\text{Phase shift} = \frac{-c}{b}$$

The phase shift is the amount that the curve is moved (displaced left or right) from its normal position.

NOTE: Phase angle is not the same as **phase shift**.

3.20 Applications of Trigonometric Graphs

3.20.1 Simple Harmonic Motion

Any object moving with constant angular velocity or moving up and down with a regular motion can be described in terms of SIMPLE HARMONIC MOTION.

The displacement, d, of an object moving with SHM, is given by:

$$d = R \sin \omega t$$

where R is the radius of the rotating object and ω is the angular velocity of the object.