CHAPTER THREE

VECTOR

After completing these tutorials, students should be able to:

- sketch the graph $\underline{r}(t)$ in 3 dimensions and its position vector
- find the tangent unit vector, $\underline{T}$ to the space curve
- find the Tangent Unit Vector, $\underline{T}$ and Normal Unit Principle Vector, $\underline{N}$ to the space curve
- determine $\underline{T}, \underline{N}, \underline{B}, \kappa$ and τ for a given space curve $\underline{r}(t)$
- Find velocity, speed, acceleration and direction of a particle
- Find displacement vector, $\underline{r}(t)$
- Find $\frac{\partial \underline{F}}{\partial x}$, $\frac{\partial \underline{F}}{\partial y}$, $\frac{\partial^2 \underline{F}}{\partial x^2}$, $\frac{\partial^2 \underline{F}}{\partial y^2}$ and $\frac{\partial^2 \underline{F}}{\partial x \partial y}$ for a given function $\underline{F}(x, y)$.

Question 1

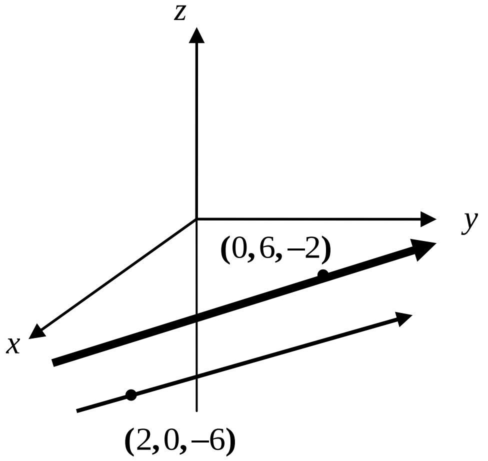
Sketch the graph $\underline{r}(t) = (2-t)\underline{i} + 3t\,\underline{j} + (2t-6)\underline{k}$ for all t in 3 dimensions.

Solution:

$\underline{r}(t) = (2-t)\underline{i} + 3t\,\underline{j} + (2t-6)\underline{k}$

$x = 2-t, y = 3t, z = 2t-6$

$t = 0, x = 2, y = 0, z = -6$
$t = 1, x = 1, y = 3, z = -4$
$t = 2, x = 0, y = 6, z = -2$

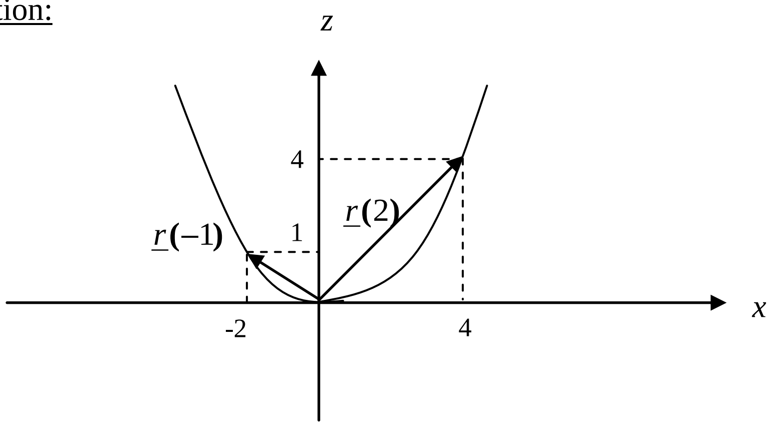


Find the line that passing through (2, 0,-6), i.e t = 0, x = 2, y = 0, z = -6 which is parallel to $u = -\underline{i} + 3t\underline{j} + 2\underline{k}$

Question 2

Sketch the graph $\underline{r}(t) = 2t\underline{i} + t^2\underline{k}$ for $-2 \le t \le 2$. Sketch also the position vector for $\underline{r}(2)$ and $\underline{r}(-1)$ on the same graph.

Solution:



$\underline{x = 2t}$ $\underline{z = t^2}$

$t = -2, x = 4, z = 4$

$t = -1, x = -2, z = 1$

$t = 0, x = 0, z = 0$

$t = 1, x = 2, z = 1$

$t = 2, x = 4, z = 4$

Question 3

Sketch the graph for the given vector function:

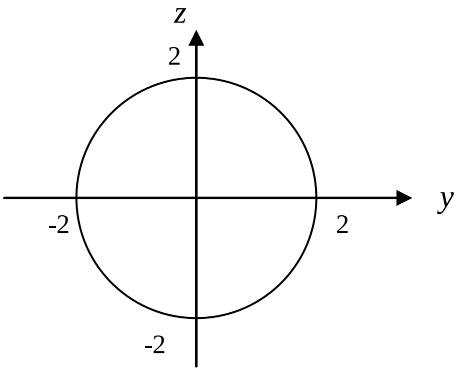
$\underline{r}(t) = 2kos\ t\underline{j} + 2\sin t\underline{k}$

Solution:

$y = 2\cos t$ $\quad\quad$ $z = 2\sin t$

$= r\cos t$ $\quad\quad$ $= r\sin t$

A circle with r = 2



Question 4

Given $\underline{F}(t) = 2\underline{i} - 5\underline{j} + t^2\underline{k}$, $\underline{G}(t) = (1-t)\underline{i} + \frac{1}{t}\underline{k}$ and $\underline{H}(t) = \sin t\underline{i} + e^t\underline{j}$.

Find the following:

(i) $2\underline{F}(t) - 3\underline{G}(t)$

Solution:

$$= 2\left(2\underline{i} - 5\underline{j} + t^2\underline{k}\right) - 3\left[(1-t)\underline{i} + \frac{1}{t}\underline{k}\right],$$

$$= (1+3t)\,\underline{i} - 10\underline{j} + \left(2t^2 - \frac{3}{t}\right)\underline{k}$$

(ii) $t^2\underline{F}(t) - 3\underline{H}(t)$

Solution:

$$= t^2\left(2\underline{i} - 5\underline{j} + t^2\underline{k}\right) - 3\left(\sin t\underline{i} + e^t\underline{j}\right)$$

$$= \left(2t^2 - 3\sin t\right)\underline{i} + \left(-5t^2 - 3e^t\right)\underline{j} + t^4\underline{k}$$

(iii) $\underline{F}(t).\underline{G}(t)$

Solution:

$$= \left(2\underline{i} - 5\underline{j} + t^2\underline{k}\right)\left((1-t)\underline{i} + \frac{1}{t}\underline{k}\right)$$

$$= 2 - t$$

(iv) $\underline{G}(t).\underline{H}(t)$

Solution:

$$= \left((1-t)\underline{i} + \frac{1}{t}\underline{k}\right)\left(\sin t\underline{i} + e^t\underline{j}\right)$$

$$= \sin t - t\sin t$$

(v) $\underline{F}(t) \times \underline{G}(t)$

Solution:

$$= \begin{vmatrix} i & j & k \\ 2 & -5 & t^2 \\ 1-t & 0 & 1/t \end{vmatrix} = \frac{-5}{t}\underline{i} - \left(\frac{2}{t} - t^2 + t^3\right)\underline{j}\left(5 - 5t\right)\underline{k}$$

(vi) $\underline{G}(t)\times\underline{H}(t)$

Solution:

$$= \begin{vmatrix} i & j & k \\ 1-t & 0 & 1/t \\ \sin t & e^t & 0 \end{vmatrix} = \frac{-e^t}{t}\,\underline{i} + \frac{\sin t}{t}\,\underline{j} + \left(e^t - te^t\right)\underline{k}$$

(vii) $2e^t + t\underline{G}(t) + 10\underline{H}(t)$

Solution:

$$2e^t + \left(t - t^2 + 10\sin t\right)\underline{i} + \left(10e^t\right)\underline{j} + \underline{k}$$

(viii) $\underline{F}(t).\left[\underline{H}(t)\times\underline{G}(t)\right]$

Solution:

$$\frac{2e^t}{t} + \frac{5\sin t}{t} - t^2e^t + t^3e^t$$

(ix) $\underline{H}(t).\left[\underline{G}(t)\times\underline{F}(t)\right]$

Solution:

$$\frac{5\sin t}{t} - t^2e^t + t^3e^t + \frac{2e^t}{t}$$

Question 5

Determine $\underline{r}'(t)$ and $\underline{r}''(t)$ for the following:

(a) $\underline{r}(t) = t\underline{i} + \left(t^2 - 1\right)\underline{j} + (3 - 5t)\underline{k}$

Solution:

$$\underline{r}'(t) = \underline{i} + 2t\,\underline{j} - 5\underline{k}$$

$$\underline{r}''(t) = 2\underline{j}$$

(b) $\underline{r}(t) = (\ln t)\underline{i} - t^{-2}\,\underline{j} + t^{-3}\,\underline{k}\,,\ t > 0$

Solution:

$$\underline{r}'(t) = \frac{1}{t}\underline{i} + 2t^{-3}\,\underline{j} - 3t^{-4}\,\underline{k}$$

$$\underline{r}''(t) = -t^{-2}\underline{i} - 6t^{-4}\,\underline{j} + 12t^{-5}\,\underline{k}$$

Question 6

Find $\underline{r}'(t)$ and determine $\underline{r}'(t)$ at the given t values:

$$\underline{r}(t) = 4kos\ t\underline{j} + 2\sin\ t\underline{k}, \quad t = \frac{3}{4}\pi$$

Solution:

$$\underline{r}'(t) = -4\sin t\underline{j} - 2kost\underline{k}$$

$$\underline{r}'(3\pi/4) = -4\sin 3\pi/4\ \underline{j} - 2kos\ 3\pi/4\ \underline{k}$$

$$= -4\left(\frac{1}{\sqrt{2}}\right)\underline{j} + 2\left(-\frac{1}{\sqrt{2}}\right)\underline{k}$$

$$= -2\sqrt{2}\underline{j} - \sqrt{2}\underline{k}$$

Question 7

If $\underline{u}(t) = t\underline{i} + t^2\underline{j} + t^3\underline{k}$ and $\underline{v}(t) = \sin t\underline{i} + \cos t\underline{j} + 2\sin t\underline{k}$, find:

(a) $\frac{d}{dt}[\underline{u}(t).\underline{v}(t)]$

Solution:

$$\frac{d}{dt}[\underline{u}(t).\underline{v}(t)]$$

$$= \frac{du(t)}{dt}.v(t) + u(t).\frac{dv(t)}{dt}$$

$$= \left(\underline{i} + 2t\underline{j} + 3t^2\underline{k}\right)(\sin t\underline{i} + \cos t\underline{j} + 2\sin t\underline{k}) +$$

$$\left(t\underline{i} + t^2\underline{j} + t^3\underline{k}\right)\left(kost\underline{i} - \sin t\underline{j} + 2\cos t\underline{k}\right)$$

$$= \left(1 + 5t^2\right)\sin t + \left(3t + 2t^3\right)\cos t$$

(b) $\frac{d}{dt}[\underline{u}(t)\times\underline{v}(t)]$

Solution:

$$\frac{d}{dt}[\underline{u}(t)\times\underline{v}(t)]$$

$$= \frac{du(t)}{dt}\times v(t) + u(t)\times\frac{dv(t)}{dt}$$

$$\frac{du(t)}{dt}\times v = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ \sin t & \cos t & 2\sin t \end{vmatrix}$$

$$= \left(4t\sin t - 3t^2\cos t\right)\underline{i} - \left(2\sin t - 3t^2\sin t\right)\underline{j} + \left(\cos t - 2t\sin t\right)\underline{k}$$

$$\underline{u}\times\frac{dv(t)}{dt} = \begin{vmatrix} i & j & k \\ t & t^2 & t^3 \\ \cos t & -\sin t & 2\cos t \end{vmatrix}$$

$$= \left(2t^2\cos t + t^3\sin t\right)\underline{i} - \left(2t\cos t - t^3\cos t\right)\underline{j} + \left(-t\sin t - t^2\cos t\right)\underline{k}$$

$$= \left(4t\sin t - 3t^2\cos t + 2t^2\cos t + t^3\sin t\right)\underline{i} + \left(-2\sin t + 3t^2\sin t - 2t\cos t + t^3\cos t\right)\underline{j} + \left(\cos t - 2t\sin t - t\sin t - t^2\cos t\right)\underline{k}$$

$$= \left(4t\sin t - t^2\cos t + t^3\sin t\right)\underline{i} + \left(-2\sin t + 3t^2\sin t - 2t\cos t + t^3\cos t\right)\underline{j} + \left(\cos t - 3t\sin t - t^2\cos t\right)\underline{k}$$

$$= \left[\left(t^3 + 4t\right)\sin t - t^2\cos t\right]\underline{i} + \left[\left(3t^2 - 2\right)\sin t + \left(t^3 - 2t\right)\cos t\right]\underline{j} + \left[-3t\sin t + \left(1 - t^2\right)\cos t\right]\underline{k}$$

Question 8

Find the tangent unit vector, $\underline{T}$ to the space curve, which is defined by the vector below:

$$\underline{r}(t) = t\sin\ t\underline{i} + tkos\ t\underline{j} + t\underline{k}$$

Solution:

$$\frac{d\underline{r}}{dt} = \left(t\cos t + \sin t\right)\underline{i} + (-t\sin t + \cos t)\underline{j} + \underline{k}$$

$$\left|\frac{d\underline{r}}{dt}\right| = \sqrt{\left(t\cos t + \sin t\right)^2 + (-t\sin t + \cos t)^2 + 1^2}$$

$$= \sqrt{t^2 + 2}$$

$$\underline{T} = \frac{1}{\sqrt{t^2 + 2}}\left[\left(t\cos t + \sin t\right)\underline{i} + (-t\sin t + \cos t)\underline{j} + \underline{k}\right]$$

Question 9

Find the Tangen Unit Vector, $\underline{T}$ and Normal Unit Principle Vector, $\underline{N}$ to the space curve, which is determined by the vector below:

$$\underline{r}(t) = e^{t}\ \underline{i} + e^{-t}\ \underline{j} + t\sqrt{2}\underline{k}$$

Solution:

$$\underline{r}(t) = e^{t}\underline{i} + e^{-t}\underline{j} + t\sqrt{2}\underline{k}$$

$$\frac{d\underline{r}}{dt} = e^{t}\underline{i} - e^{-t}\underline{j} + \sqrt{2}\underline{k}$$

$$\left|\frac{d\underline{r}}{dt}\right| = \sqrt{\left(e^{t}\right)^{2} - \left(e^{-t}\right)^{2} + \left(\sqrt{2}\right)^{2}}$$

$$= \sqrt{e^{2t} - e^{-2t} + 2}$$

$$= \sqrt{\left(e^{t} + e^{-t}\right)^{2}}$$

$$= e^{t} + e^{-t}$$

$$\underline{T} = \frac{\frac{d\underline{r}}{dt}}{\left|\frac{d\underline{r}}{dt}\right|} = \frac{e^{t}\underline{i} - e^{-t}\underline{j} + \sqrt{2}\underline{k}}{e^{t} + e^{-t}}$$

$$\frac{d\underline{T}}{dt} = \left(\frac{\left(e^{t} + e^{-t}\right)\left(e^{t}\underline{i} + e^{-t}\underline{j}\right) - \left(e^{t}\underline{i} - e^{-t}\underline{j} + \sqrt{2}\underline{k}\right)\left(e^{t} - e^{-t}\right)}{\left(e^{t} + e^{-t}\right)^{2}}\right)$$

$$= \left(\frac{2i + 2j + \sqrt{2}\left(e^{-t} - e^{t}\right)\underline{k}}{\left(e^{t} + e^{-t}\right)^{2}}\right)$$

$$\left|\frac{d\underline{T}}{dt}\right| = \sqrt{\frac{4 + 4 + 2\left(e^{-t} - e^{t}\right)^{2}}{\left(e^{t} + e^{-t}\right)^{4}}}$$

$$= \sqrt{\frac{2}{\left(e^{t} + e^{-t}\right)^{2}}}$$

$$= \frac{\sqrt{2}}{e^{t} + e^{-t}}$$

$$\underline{N} = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = \left(\frac{2i + 2j + \sqrt{2}\left(e^{-t} - e^{t}\right)\underline{k}}{\left(e^{t} + e^{-t}\right)^{2}} \right) \div \frac{\sqrt{2}}{e^{t} + e^{-t}}$$

$$= \frac{1}{e^{t} + e^{-t}}\left(2i + 2j - \left(e^{t} - e^{-t}\right)\underline{k}\right)$$

Question 10

Find the integration value of :

$$\int_{0}^{\frac{\pi}{4}} (\sin t\underline{i} - kost\underline{j} + \tan t\underline{k})dt$$

Solution:

$$\int_{0}^{\frac{\pi}{4}} (\sin t\underline{i} - kost\underline{j} + \tan t\underline{k})dt$$

$$= \left[-kost\right]_{0}^{\pi/4}\underline{i} - \left[\sin t\right]_{0}^{\pi/4}\underline{j} + \int_{0}^{\pi/4} \frac{\sin t}{\cos t}dt$$

$$= \left[-kos\,\pi/4 - (-kos0)\right]\underline{i} - \left[\sin \pi/4 - \sin 0\right]\underline{j} + \left[\ln\left|\cos t\right|\right]_{0}^{\pi/4}$$

$$= \left[-\frac{1}{\sqrt{2}} + 1\right]\underline{i} - \left[\frac{1}{\sqrt{2}} - 0\right]\underline{j} + \left[-\ln\left|\cos \pi/4\right| - \left(-\ln\left|\cos 0\right|\right)\right]\underline{k}$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right)\underline{i} - \frac{1}{\sqrt{2}}\underline{j} + \left(\frac{1}{2}\ln 2\right)\underline{k}$$

Question 11

Determine $\underline{T}, \underline{N}, \underline{B}, \kappa$ and τ for space curve $\underline{r}(t)$ below:

(a) $\underline{r}(t) = (3\sin t)\underline{i} + (3\cos t)\underline{j} + 4t\underline{k}$

Solution:

$$\frac{\partial \underline{r}}{\partial t} = (3\cos t)\underline{i} - (3\sin t)\underline{j} + 4\underline{k}$$

$$\left|\frac{\partial \underline{r}}{\partial t}\right| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (4)^2}$$

$$= \sqrt{9(\cos^2 t + \sin^2 t) + 16}$$

$$= \sqrt{9+16}$$

$$= \sqrt{25}$$

$$= 5$$

$$\underline{T} = \frac{\frac{\partial \underline{r}}{\partial t}}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{(3\cos t)\underline{i} - (3\sin t)\underline{j} + 4\underline{k}}{5} = \frac{1}{5}[(3\cos t)\underline{i} - (3\sin t)\underline{j} + 4\underline{k}]$$

$$\frac{\partial \underline{T}}{\partial t} = \frac{1}{5}[(-3\sin t)\underline{i} - (3\cos t)\underline{j}]$$

$$\left|\frac{\partial \underline{T}}{\partial t}\right| = \sqrt{\left(\frac{-3}{5}\sin t\right)^2 + \left(\frac{-3}{5}\cos t\right)^2}$$

$$= \sqrt{\frac{9}{25}(\sin^2 t + \cos^2 t)}$$

$$= \frac{3}{5}$$

$$\underline{N} = \frac{\frac{\partial \underline{T}}{\partial t}}{\left|\frac{\partial \underline{T}}{\partial t}\right|} = \frac{\frac{1}{5}[(-3\sin t)\underline{i} - (3\cos t)\underline{j}]}{\frac{3}{5}}$$

$$= \frac{1}{3}[(-3\sin t)\underline{i} - (3\cos t)\underline{j}]$$

$$= -\sin t\,\underline{i} - \cos t\,\underline{j}$$

$$\underline{B} = \underline{T} \times \underline{N}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{3}{5}\mathbf{cos}\,t & \frac{-3}{5}\mathbf{sin}\,t & \frac{4}{5} \\ -\mathbf{sin}\,t & -\mathbf{cos}\,t & 0 \end{vmatrix}$$

$$= \frac{4}{5}\mathbf{cos}\,t\,\underline{i} - \frac{4}{5}\mathbf{sin}\,t\underline{j} - \frac{3}{5}\underline{k}$$

$$\frac{\partial \underline{B}}{\partial t} = -\frac{4}{5}\mathbf{sin}\,t\,\underline{i} - \frac{4}{5}\mathbf{cos}\,t\underline{j}$$

$$\left|\frac{\partial \underline{B}}{\partial t}\right| = \sqrt{\left(-\frac{4}{5}\mathbf{sin}\,t\right)^2 + \left(-\frac{4}{5}\mathbf{cos}\,t\right)^2}$$

$$= \sqrt{\frac{16}{25}\left(\mathbf{sin}^2\,t + \mathbf{cos}^2\,t\right)}$$

$$= \frac{4}{5}$$

$$\text{k} = \frac{\left|\frac{\partial \underline{T}}{\partial t}\right|}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{\frac{3}{5}}{5} = \frac{3}{25}$$

$$\text{t} = \frac{\left|\frac{\partial \underline{B}}{\partial t}\right|}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{4/5}{5} = \frac{4}{25}$$

(b) $\underline{r}(t) = \left(e^t\,\mathbf{cos}\,t\right)\underline{i} + \left(e^t\,\mathbf{sin}\,t\right)\underline{j} + 2\underline{k}$

<u>Solution:</u>

$$\frac{\partial \underline{r}}{\partial t} = (-e^t\,\mathbf{sin}\,t + e^t\,\mathbf{cos}\,t)\,\underline{i} + (e^t\,\mathbf{cos}\,t + e^t\,\mathbf{sin}\,t)\,\underline{j}$$

$$\left|\frac{\partial \underline{r}}{\partial t}\right| = \sqrt{(-e^t\,\mathbf{sin}\,t + e^t\,\mathbf{cos}\,t)^2 + (-(e^t\,\mathbf{cos}\,t + e^t\,\mathbf{sin}\,t))^2}$$

$$= \sqrt{e^{2t}(\mathbf{sin}^2\,t + \mathbf{cos}^2\,t) + e^{2t}(\mathbf{sin}^2\,t + \mathbf{cos}^2\,t)}$$

$$= \sqrt{e^{2t} + e^{2t}}$$

$$= \sqrt{2e^{2t}}$$

$$= \sqrt{2}e^t$$

$$\underline{T} = \frac{\frac{\partial \underline{r}}{\partial t}}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{(-e^t \sin t + e^t \cos t)\underline{i} + (e^t \cos t + e^t \sin t)\underline{j}}{\sqrt{2}e^t}$$

$$= \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\underline{i} + (\cos t + \sin t)\underline{j}]$$

$$\frac{\partial \underline{T}}{\partial t} = \frac{1}{\sqrt{2}}[(-\cos t - \sin t)\underline{i} + (-\sin t + \cos t)\underline{j}]$$

$$\left|\frac{\partial \underline{T}}{\partial t}\right| = \sqrt{\frac{1}{2}(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2}$$

$$= \sqrt{\frac{1}{2}\left(\cos^2 t + \sin^2 t + 2\sin t \cos t + \sin^2 t + \cos^2 t - 2\sin t \cos t\right)}$$

$$= \sqrt{1}$$

$$= 1$$

$$\underline{N} = \frac{\frac{\partial \underline{T}}{\partial t}}{\left|\frac{\partial \underline{T}}{\partial t}\right|} = \frac{\frac{1}{\sqrt{2}}[(-\cos t - \sin t)\underline{i} + (-\sin t + \cos t)\underline{j}]}{1}$$

$$= -\frac{1}{\sqrt{2}}[(\cos t + \sin t)\underline{i} + (\sin t - \cos t)\underline{j}]$$

$$\underline{B} = \underline{T} \times \underline{N}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{1}{\sqrt{2}}(-\cos t - \sin t) & \frac{1}{\sqrt{2}}(-\sin t + \cos t) & 0 \\ -\frac{1}{\sqrt{2}}(\cos t + \sin t) & -\frac{1}{\sqrt{2}}(\sin t - \cos t) & 0 \end{vmatrix}$$

$$= 0\underline{i} - 0\underline{j} + \left[\frac{1}{2}(\cos t - \sin t)(\cos t - \sin t) - \left(-\frac{1}{2}(\cos t + \sin t)^2\right)\right]\underline{k}$$

$$= \frac{1}{2}\left[(\cos t - \sin t)^2 + (\cos t + \sin t)^2\right]\underline{k}$$

$$= \frac{1}{2}\left[2\cos^2 t + 2\sin^2 t\right]\underline{k}$$

$$= \underline{k}$$

$$\frac{\partial \underline{B}}{\partial t} = 0\underline{k} = 0$$

$$\left|\frac{\partial \underline{B}}{\partial t}\right| = \sqrt{0^2}$$

$$= 0$$

$$\text{k} = \frac{\left|\frac{\partial \underline{T}}{\partial t}\right|}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{1}{\sqrt{2}e^t}$$

$$\text{t} = \frac{\left|\frac{\partial \underline{B}}{\partial t}\right|}{\left|\frac{\partial \underline{r}}{\partial t}\right|} = \frac{0}{\sqrt{2}e^t} = 0$$

Question 12

Given that displacement vector ,

$$\underline{r}(t) = \left(3\cos t\right)\underline{i} + \left(3\sin t\right)\underline{j} + t^2\underline{k}$$

Find velocity, speed, acceleration and direction of particle at $t = 2$.

<u>Solution:</u>

$$\underline{v} = \frac{\partial r}{\partial t} = -3\sin t\,\underline{i} + 3\cos t\underline{j} + 2t\underline{k}$$

Velocity, $\underline{v}\,|_{t=2} = -3\sin 2\,\underline{i} + 3\cos 2\,\underline{j} + 4\underline{k}$

$$|\underline{v}| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2t)^2}$$
$$= \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$$
$$= \sqrt{9 + 4t^2}$$

Speed, $|\underline{v}|_{t=2} = \sqrt{9 + 4(4)} = \sqrt{25} = 5$

$$\underline{a} = \frac{\partial v}{\partial t} = -3\cos t\,\underline{i} - 3\sin t\underline{j} + 2\underline{k}$$

Acceleration, $\underline{a}\,|_{t=2} = -3\cos 2\,\underline{i} - 3\sin 2\,\underline{j} + 2\underline{k}$

Direction, $E_{\underline{v}}\,|_{t=2} = \dfrac{\underline{v}}{|\underline{v}|} = \dfrac{-3\sin 2\,\underline{i} + 3\cos 2\,\underline{j} + 4\underline{k}}{5}$

$$= \frac{1}{5}\left(-3\sin 2\,\underline{i} + 3\cos 2\,\underline{j} + 4\underline{k}\right)$$

Question 13

Given that velocity, $\underline{v}(t) = -3\underline{i} + \sin^2 t\,\underline{j} + \sin^2 t\underline{k}$.

Find displacement vector, $\underline{r}(t)$ if $\underline{r}(0) = \underline{j}$.

<u>Solution:</u>

$$\underline{v}(t) = \frac{\partial r}{\partial t} = -3\underline{i} + \sin^2 t\,\underline{j} + \sin^2 t\underline{k}$$

$$\underline{r}(t) = \int(-3\underline{i} + \sin^2 t\,\underline{j} + \sin^2 t\underline{k})\partial t$$

$$= -3t\,\underline{i} + \int(\sin^2 t)dt\underline{j} + \int(\sin^2 t)\partial t\underline{k}$$

$$= -3t\,\underline{i} + \frac{1}{2}\left[t - \frac{\sin 2t}{2} + c_1\right]\underline{j} + \frac{1}{2}\left[t - \frac{\sin 2t}{2} + c_2\right]\underline{k}$$

Given that $\underline{r}(0) = \underline{j}$, therefore

$$\underline{r}(0) = 0\underline{i} + \frac{1}{2}[0-0+c_1]\,\underline{j} + \frac{1}{2}[0-0+c_2]\,\underline{k} = \frac{1}{2}c_1\underline{j} + \frac{1}{2}c_2\underline{k} = \underline{j}$$

$\Rightarrow c_1 = 2$ and $c_2 = 0$

$$\therefore\ \underline{r}(t) = -3t\,\underline{i} + \frac{1}{2}\left[t - \frac{\mathbf{sin}\,2t}{2} + 2\right]\underline{j} + \frac{1}{2}\left[t - \frac{\mathbf{sin}\,2t}{2}\right]\underline{k}$$

$$= -3t\,\underline{i} + \frac{1}{4}[2t - \mathbf{sin}\,2t + 4]\,\underline{j} + \frac{1}{4}[2t - \mathbf{sin}\,2t]\,\underline{k}$$

Question 14

Given that vector function,

$$\underline{F}(x,y) = \left(2x^2y - x^4\right)\underline{i} + \left(e^{xy} - y\sin x\right)\underline{j} + \left(x^2 kosy\right)\underline{k}\ .$$

Find:

(a) $\dfrac{\partial \underline{F}}{\partial x}$

Solution:

$$\frac{\partial \underline{F}}{\partial x} = F_x = \left(4xy - 4x^3\right)\underline{i} + \left(ye^{xy} - y\,\mathbf{cos}\,x\right)\underline{j} + \left(2x\,\mathbf{cos}\,y\right)\underline{k}$$

(b) $\dfrac{\partial \underline{F}}{\partial y}$

Solution:

$$\frac{\partial \underline{F}}{\partial y} = F_y = 2x^2\,\underline{i} + \left(xe^{xy} - \mathbf{sin}\,x\right)\underline{j} + \left(-x^2\,\mathbf{sin}\,y\right)\underline{k}$$

(c) $\dfrac{\partial^2 \underline{F}}{\partial x^2}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial x^2} = F_{xx} = (4y - 12x^2)\,\underline{i} + \left(y^2e^{xy} + y\,\mathbf{sin}\,x\right)\underline{j} + \left(2\,\mathbf{cos}\,y\right)\underline{k}$$

(d) $\dfrac{\partial^2 \underline{F}}{\partial y^2}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial y^2} = F_{yy} = \left(x^2 e^{xy}\right)\underline{i} - \left(x^2 \cos y\right)\underline{k}$$

(e) $\dfrac{\partial^2 \underline{F}}{\partial x \partial y}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial x \partial y} = F_{xy} = 4x\underline{i} + \left(xye^{xy} + e^{xy} - \cos x\right)\underline{j} - \left(2x \sin y\right)\underline{k}$$

Question 15

Given that vector function,

$$\underline{F}(x,y) = \left(2xy\right)\underline{i} + \left(x^2 - 2y\right)\underline{j} + \left(x + y^2\right)\underline{k}\ .$$

Find:

(a) $\dfrac{\partial \underline{F}}{\partial x}$

Solution:

$$\frac{\partial \underline{F}}{\partial x} = F_x = 2y\underline{i} + 2x\underline{j} + \underline{k}$$

(b) $\dfrac{\partial \underline{F}}{\partial y}$

Solution:

$$\frac{\partial \underline{F}}{\partial y} = F_y = 2x\underline{i} - 2\underline{j} + 2y\underline{k}$$

(c) $\dfrac{\partial^2 \underline{F}}{\partial x^2}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial x^2} = F_{xx} = 2\underline{j}$$

(d) $\dfrac{\partial^2 \underline{F}}{\partial y^2}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial y^2} = F_{yy} = 2\underline{k}$$

(e) $\dfrac{\partial^2 \underline{F}}{\partial x \partial y}$

Solution:

$$\frac{\partial^2 \underline{F}}{\partial x \partial y} = F_{xy} = 2\underline{i}$$