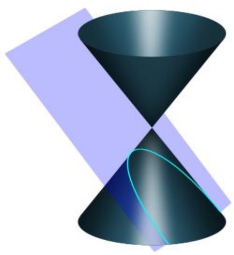
CHAPTER FOUR

ANALYTIC GEOMETRY

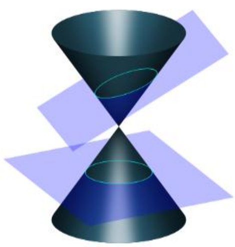
4.0 Introduction

Analytic geometry is a union of geometry and algebra. It enables us to interpret certain algebraic relationships geometrically. It also enables us to analyze certain geometric concepts algebraically. In this chapter, we discuss analytic geometry: conic sections.

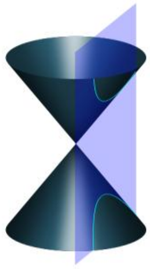
Conic sections are curves which are obtained by intersecting a right circular cone(double cone) with a plane.



The intersection between a right circular conical surface and a plane that cuts through both halves of the cone creates hyperbola.



The intersection of a right circular conical surface and a plane perpendicular to the axis creates a circle. If the plane is not perpendicular to the axis, then the intersection produces ellipse.



The intersection of a right circular conical surface and a plane parallel to a generating straight line of that surface produces parabola.

To identify conics of the equation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

If $B^2 - 4AC > 0$ ➧ hyperbola

If $B^2 - 4AC = 0$ ➧ parabola

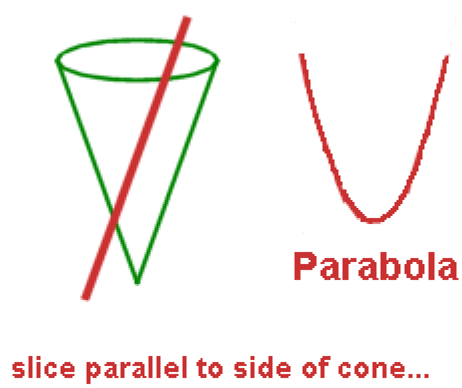
If $B^2 - 4AC < 0$ ➧ ellipse

The parabola has many applications in situations where radiation needs to be concentrated at one point (e.g. radio telescopes, solar radiation collectors) or transmitted from a single point into a wide parallel beam (e.g. headlight reflectors).

The path that orbiting satellites trace out is the ellipse. A lot of architectural designs (especially bridges) use the ellipse as a pleasing (and strong) shape.

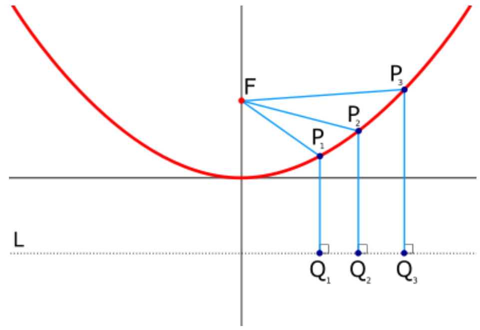
One property of ellipses is that a sound (or any radiation) beginning in one focus of the ellipse will be reflected so it can be heard clearly at the other focus.

4.1 The Parabola

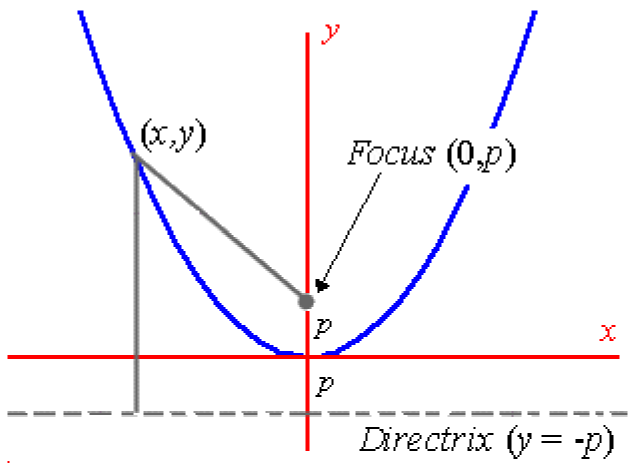


The parabola is defined as the locus of points(P_1, P_2, P_3,…, P_n) which move so that it is always the same distance from a fixed point (called the focus, F) and a given line (called the directrix, L).

$$FP_1 = P_1Q_1$$



4.1.1 The Formula for a Parabola



Using the distance formula, we have

$$\sqrt{(x-0)^2+(y-p)^2}=y+p$$

Squaring both sides gives:

$$(x-0)^2+(y-p)^2=(y+p)^2$$

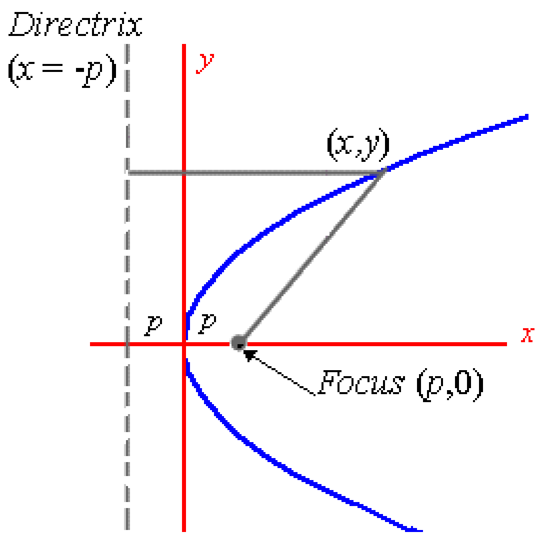
Simplifying gives us the formula for a parabola:

$$x^2=4py$$

In more familiar form, we can write this as:

$$y = \frac{x^2}{4p}$$

We can also have the situation where the axis of the parabola is horizontal:



In this case, we have the *relation:*

$$y^2 = 4px$$

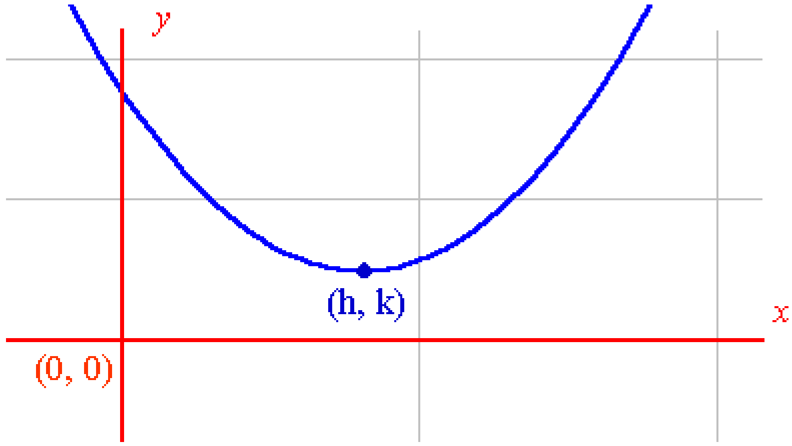
4.1.2 Shifting the Vertex of a Parabola from the Origin

This is similar to the case when we shifted the centre of a circle from the origin.

To shift the vertex of a parabola from (0, 0) to (h, k), each x in the equation becomes (x - h) and each y becomes (y - k).

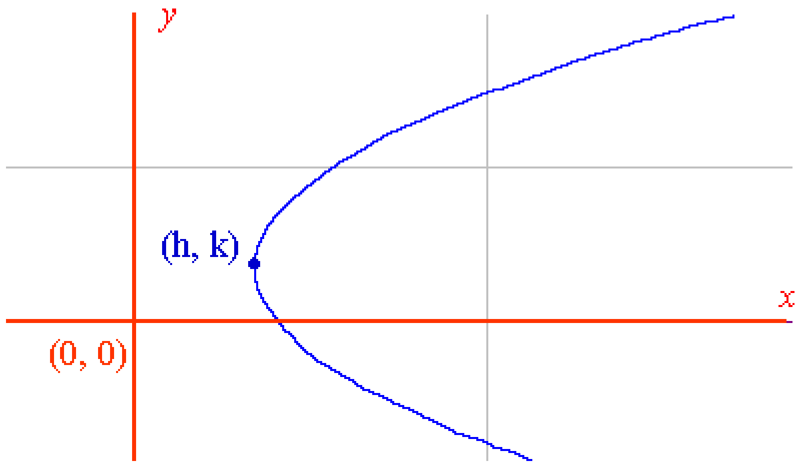
So if the axis of a parabola is **vertical**, and the vertex is at (h, k), we have

$$(x-h)^2 = 4p(y-k)$$

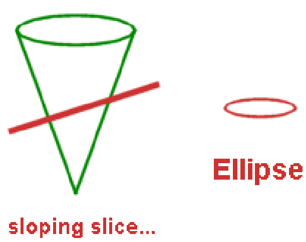


If the axis of a parabola is **horizontal**, and the vertex is at (*h*, *k*), the equation becomes

$(y-k)^2 = 4p(x-h)$.

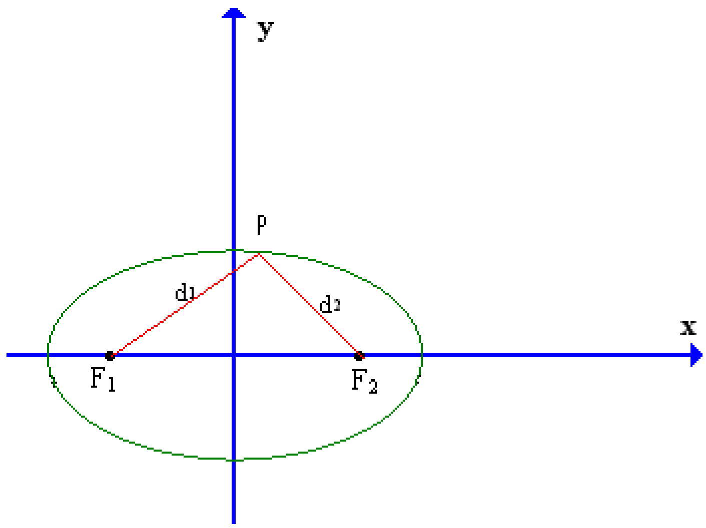


4.2 The Ellipse

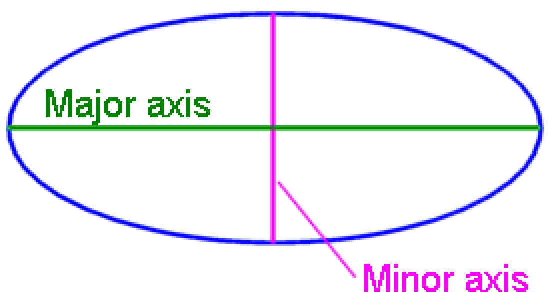


The ellipse is defined as the locus of points P(x,y) which move so that the sum of its distances from two fixed points F_1 and F_2 (called foci) is constant.

$$d_1 + d_2 = constant$$



4.2.1 Ellipse With Horizontal Major Axis



The equation for an ellipse with a horizontal major axis is given by:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The ellipse is defined as the locus of a point (x, y) which moves so that the sum of its distances from two fixed points (called foci) is constant.

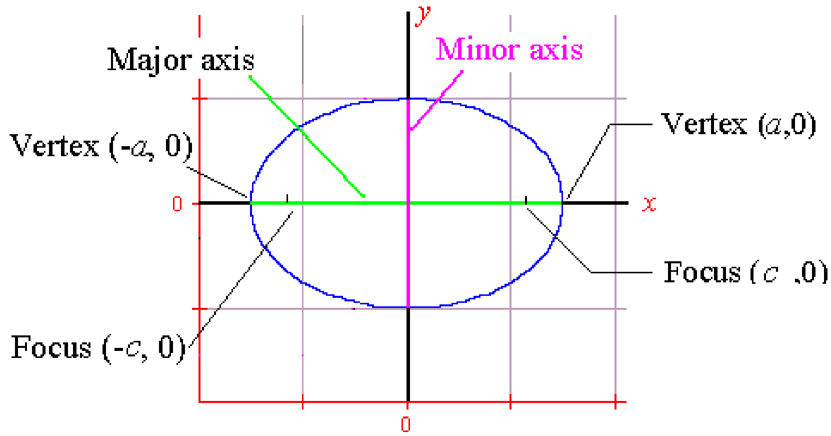
We can produce an ellipse by pinning the ends of a piece of string and keeping a pencil tightly within the boundary of the string.

An ellipse is a stretched circle.

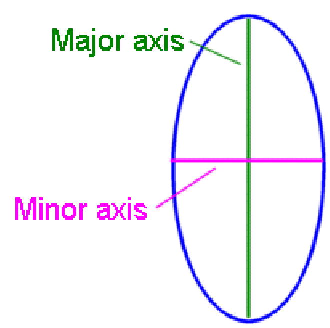
The **foci** (plural of 'focus') of an ellipse (with horizontal major axis) are at (-*c*,0) and (*c*,0), where *c* is given by:

$$c = \sqrt{a^2 - b^2}$$

The **vertices** of an ellipse are at (-*a*, 0) and (*a*, 0).



4.2.2 Vertical Major Axis



NOTE: Our first example above had a horizontal major axis. If the major axis is **vertical**, then the formula becomes:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

We always choose our a and b such that $a > b$. The major axis is always associated with a.

4.2.3 Ellipses with Centre not at the Origin

Like the other conics, we can move the ellipse so that its axes are not on the x-axis and y-axis.

For the horizontal major axis case, if we move the intersection of the major and minor axes to the point (h, k), we have:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Theorem:

The equation

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

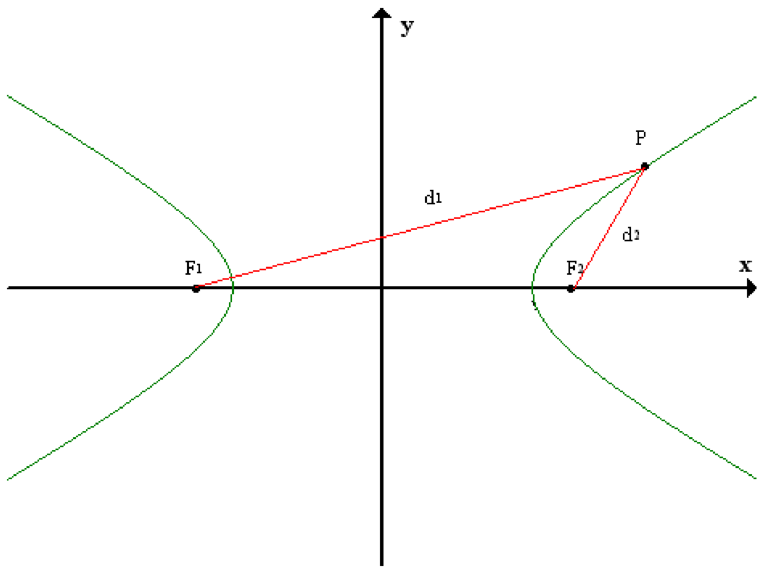
with $a > b > 0$ describes an ellipse with foci at $(x_0 - c, y_0)$ and $(x_0 + c, y_0)$, where $c = \sqrt{a^2 - b^2}$. The center of the ellipse is at the point (x_0, y_0) and the vertices are located at $(x_0 \pm a, y_0)$ on the major axis. The endpoints of the minor axis are located at $(x_0, y_0 \pm b)$.

The equation

$$\frac{(x-x_0)^2}{b^2} + \frac{(y-y_0)^2}{a^2} = 1$$

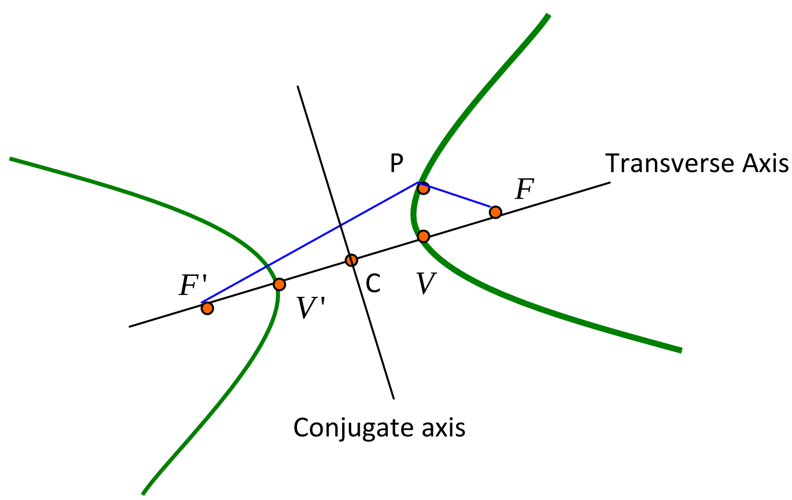
with $a > b > 0$ describes an ellipse with foci at $(x_0, y_0 - c)$ and $(x_0, y_0 + c)$, where $c = \sqrt{a^2 - b^2}$. The center of the ellipse is at the point (x_0, y_0) and the vertices are located at $(x_0, y_0 \pm a)$ on the major axis. The endpoints of the minor axis are located at $(x_0 \pm b, y_0)$.

4.3 The Hyperbola



$$|d_1 - d_2| = constant$$

The hyperbola is defined as the set of all points P such that the absolute value of the difference of the distances of P to two fixed points in the plane is a positive constant.



$$|d_1 - d_2| = \text{constant}$$

Each of the fixed points, F and F' is called a focus. The intersection points V and V' of the line through the foci and the two branches of the hyperbola are called vertices, and each is called a vertex. The line segment $V'V$ is called the transverse axis. The midpoint of the transverse axis is the center of the hyperbola.

Theorem:

The equation

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

describes a hyperbola with foci at the points $(x_0 - c, y_0)$ and $(x_0 + c, y_0)$ where $c = \sqrt{a^2 + b^2}$.
The center of the hyperbola is at the point (x_0, y_0) and the vertices are located at $(x_0 \pm a, y_0)$.
The asymptotes are $y = \pm\frac{b}{a}(x - x_0) + y_0$.

The equation

$$\frac{(y-y_0)^2}{a^2} - \frac{(x-x_0)^2}{b^2} = 1$$

describes a hyperbola with foci at the points $(x_0, y_0 - c)$ and $(x_0, y_0 + c)$ where $c = \sqrt{a^2 + b^2}$.
The center of the hyperbola is at the point (x_0, y_0) and the vertices are located at $(x_0, y_0 \pm a)$.
The asymptotes are $y = \pm\frac{a}{b}(x - x_0) + y_0$.

4.4 Standard Equations of Parabolas, Ellipses and Hyperbolas.

Parabolas	
$(x-h)^2 = 4a(y-k)$	$(y-k)^2 = 4a(x-h)$
Vertex (h, k)	Vertex (h, k)
Focus (h, k+a)	Focus (h+a, k)
If a < 0, then curve opens down	If a < 0, then curve opens left
If a > 0, then curve opens up	If a > 0, then curve opens right

Ellipses ; a > b > 0 ; $c = \sqrt{a^2 - b^2}$	
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Center(h, k)	Center(h, k)
Major axis is horizontal, length=2a	Major axis is vertical, length=2a
Minor axis is vertical, length=2b	Minor axis is horizontal, length=2b
Foci(h-c, k) and (h+c, k)	Foci (h, k-c) and (h, k+c)

Hyperbolas	
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$	$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$
Center(h, k)	Center(h, k)
Transverse axis is horizontal, length=2a	Transverse axis is vertical, length=2a
Conjugate axis is vertical, length=2b	Conjugate axis is horizontal, length=2b
asymptotes, (y-k)/(x-h) = ± b/a	asymptotes, (y-k)/(x-h) = ± a/b
Vertices, (h±a, k)	Vertices, (h, k±a)