4.4 Gauss-Seidel method ....

The Gauss-Seidel method is the most commonly used iterative method, which employs initial guesses and then iterates to obtain refined estimates of the solution. The Gauss-Seidel method is particularly well-suited for large numbers of equations.

A matrix **A** of dimension N×N is said to be **strictly diagonally dominant** provided that

$$|a_{KK}| > \sum_{\substack{j=1 \\ j\neq k}}^{N} |a_{kj}| \quad \text{for } k = 1, 2, \ldots, N$$

This means that in each row of the matrix the magnitude of the element on the main diagonal must exceed the sum of the magnitudes of all other elements in the row.

For example, given linear system

$$\begin{aligned} 8x_1 + x_2 - x_3 &= 8 \\ x_1 - 7x_2 + 2x_3 &= -4 \\ 2x_1 + x_2 + 9x_3 &= 12 \end{aligned}$$

The coefficient matrix of the linear system above is strictly diagonally dominant because:

In row 1 : $|8| > |1| + |-1|$

In row 2 : $|-7| > |1| + |2|$

In row 3 : $|9| > |2| + |1|$

Gauss-Seidel iteration – To solve the linear system **AX=b** by starting with the initial guess **X** = P_0 =0 and generating a sequence { P_k} for k = 1, 2, 3, ... k that converges to the solution. A sufficient condition for the method to be applicable is that **A** is strictly dominant.

In the Gauss-Seidel iteration method, the linear system

AX=b

Or

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

can be written as

$$x_1^{(k+1)} = \frac{b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - \ldots - a_{1n}x_n^{(k)}}{a_{11}}$$

$$x_2^{(k+1)} = \frac{b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)} - \ldots - a_{2n}x_n^{(k)}}{a_{22}}$$

or in the general form

$$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}} \qquad \forall i = 1, 2, 3, ..., n$$

The process is repeated until *k*th iteration while

$$\left\| x^{(k)} - x^{(k-1)} \right\|_\infty = \max_{1 \le i \le n} \left\{ \left| x_i^{(k)} - x_i^{(k-1)} \right| \right\} < \varepsilon$$

where error tolerance ε is given.

Example:

Use the Gauss-Seidel method to obtain the solution of the linear system

$$\begin{aligned} 12x_1 + 3x_2 - \quad x_3 &= 15 \\ x_1 + 8x_2 + \quad x_3 &= 20 \\ 2x_1 - x_2 + 10x_3 &= 30 \end{aligned}$$

accurate to within ε = 0.001.

$$x_1^{(k+1)} = \frac{15 - 3x_2^{(k)} + x_3^{(k)}}{12}$$

$$x_2^{(k+1)} = \frac{20 - x_1^{(k+1)} - x_3^{(k)}}{8}$$

$$x_3^{(k+1)} = \frac{30 - 2x_1^{(k+1)} + x_2^{(k+)}}{10}$$

Assume $\mathbf{X}^{(0)} = \mathbf{0}$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\Vert x^{(k)} - x^{(k-1)} \right\Vert_\infty$
0	0	0	0	
1	1.25	2.3438	2.9844	2.9844
2	0.9128	2.0129	3.0187	0.3372
3	0.9983	1.9979	3.0001	0.0855
4	1.0005	1.9999	2.9999	0.0022
5	1.0000	2.0000	3.0000	0.0005

$\therefore$ $\mathbf{X} = (x_1, x_2, x_3)^T = (1.0000, 2.0000, 3.0000)$

Example :

Use Gauss elimination to solve

$$2x_1 + x_2 - x_3 = 4$$
$$4x_1 + x_2 + 2x_3 = 10$$
$$3x_1 + x_2 + x_3 = 7.5$$

Gauss Elimination

Operation	multiple	$a_{i1}(x_1)$	$a_{i2}(x_2)$	$a_{i3}(x_3)$	b_i
		2	1	–1	4
		4	1	2	10
		3	1	1	7.5
		2	1	–1	4
$R_2+m_1R_1$	$m_1 = -\frac{a_{21}}{a_{11}} = \frac{-4}{2} = -2$	0	–1	4	2
$R_3+m_2R_1$	$m_2 = -\frac{a_{31}}{a_{11}} = \frac{-3}{2}$	0	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
		2	1	–1	4
		0	–1	4	2
$R_3+m_3R_2$	$m_3 = -\frac{a_{32}}{a_{22}} = -\frac{-1/2}{-1} = -\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$

Back substitution: $2x_1 + x_2 - x_3 = 4$

$$-x_2 + 4x_3 = 2$$

$$\frac{1}{2}x_3 = \frac{1}{2}$$

$\therefore x_3 = \frac{1/2}{1/2} = 1$ $\quad\quad$ $-x_2 + 4(1) = 2 \quad \therefore x_2 = 2$

$2x_1 + 2 - 1 = 4 \quad \therefore x_1 = 1.5$

Example :

Use the methods of Gauss elimination to solve

$$10x_1 - 7x_2 = 7$$

$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$

$$5x_1 - x_2 + 5x_3 = 6$$

Gauss Elimination

Operation	multiple	$a_{i1}(x_1)$	$a_{i2}(x_2)$	$a_{i3}(x_3)$	b_i
		10	**-7**	**0**	**7**
		-3	**2.099**	**6**	**3.901**
		5	**-1**	**5**	**6**
		10	-7	0	7
$R_2+m_1R_1$	$m_1 = -\frac{a_{21}}{a_{11}} = -\frac{-3}{10} = 0.3$	0	-0.001	6	6.001
$R_3+m_2R_1$	$m_2 = -\frac{a_{31}}{a_{11}} = -\frac{5}{10} = -0.5$	0	2.5	5	2.5
		10	-7	0	7
		0	-0.001	6	6.001
$R_3+m_3R_2$	$m_3 = -\frac{a_{32}}{a_{22}} = -\frac{2.5}{-0.001} = 2500$	0	0	15005	15004

Back substitution:

$15005x_3 = 15004$ $\quad\therefore x_3 = 0.99993$

$-0.001x_2 + 6x_3 = 6.001$ $\quad\therefore x_2 = -1.5$

$10x_1 - 7x_2 = 7$ $\quad\therefore x_1 = -0.3500$

Hence the solution is $x_1 = -0.35, x_2 = -1.5, x_3 = 0.99993$

Compare with the exact solution of $x_1 = 0, x_2 = -1, x_3 = 1$

Gauss elimination with partial pivoting

Operation	multiple	$a_{i1}(x_1)$	$a_{i2}(x_2)$	$a_{i3}(x_3)$	b_i
		10	**-7**	**0**	**7**
		-3	**2.099**	**6**	**3.901**
		5	**-1**	**5**	**6**
		10	-7	0	7
$R_2+m_1R_1$	$m_1 = -\frac{a_{21}}{a_{11}} = -\frac{-3}{10} = 0.3$	0	-0.001	6	6.001
$R_3+m_2R_1$	$m_2 = -\frac{a_{31}}{a_{11}} = -\frac{5}{10} = -0.5$	0	2.5	5	2.5
		10	-7	0	7
$R_2 \leftrightarrow R_3$		0	2.5	5	2.5
		0	-0.001	6	6.001
		10	-7	0	7
		0	2.5	5	2.5
$R_3+m_3R_2$	$m_3 = -\frac{a_{32}}{a_{22}} = -\frac{-0.001}{2.5} = 0.00004$	0	0	6.002	6.002

Back substitution:

$6.002x_3 = 6.002 \qquad \therefore x_3 = 1$

$2.5x_2 + 5x_3 = 2.5 \qquad \therefore x_2 = -1$

$10x_1 - 7x_2 = 7 \qquad \therefore x_1 = 0$

This, in fact, is the exact solution.

LU Decomposition
AX=b

A=LU

In matrix form (matrix 3×3), this is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21}u_{11} & \ell_{21}u_{12} + u_{22} & \ell_{21}u_{13} + u_{23} \\ \ell_{31}u_{11} & \ell_{31}u_{12} + \ell_{32}u_{22} & \ell_{31}u_{13} + \ell_{32}u_{23} + u_{33} \end{bmatrix}$$

where

$a_{11} = u_{11}$

$a_{12} = u_{12}$

$a_{13} = u_{13}$

$a_{21} = l_{21}u_{11}$

$a_{22} = l_{21}u_{12} + u_{22}$

$a_{23} = l_{21}u_{13} + u_{23}$

$a_{31} = l_{31}u_{11}$

$a_{32} = l_{31}u_{12} + l_{32}u_{22}$

$a_{33} = l_{31}u_{13} + l_{32}u_{23} + u_{23}$

Example :

Use the factorization A=LU to solve the system

$$-2x_1 - 3x_2 + 4x_3 = 1$$
$$-2x_1 + 2x_2 - 3x_3 = 2$$
$$x_1 + 2x_2 - x_3 = -3$$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A=\begin{bmatrix} -2 & -3 & 4 \\ 2 & 2 & -3 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ \ell_{21}u_{11} & \ell_{21}u_{12}+u_{22} & \ell_{21}u_{13}+u_{23} \\ \ell_{31}u_{11} & \ell_{31}u_{12}+\ell_{32}u_{22} & \ell_{31}u_{13}+\ell_{32}u_{23}+u_{33} \end{bmatrix}$$

- $-2 = u_{11}$
- $-3 = u_{12}$
- $4 = u_{13}$
- $2 = l_{21}u_{11} \therefore l_{21} = -1$
- $2 = l_{21}u_{12} + u_{22} \rightarrow 2 = 3 + u_{22} \quad \therefore -1 = u_{22}$
- $-3 = l_{21}u_{13} + u_{23} \rightarrow -3 = -4 + u_{23} \quad \therefore 1 = u_{23}$
- $1 = l_{31}u_{11} \quad \therefore -1/2 = l_{31}$
- $2 = l_{31}u_{12} + l_{32}u_{22} \rightarrow 2 = 3/2 + l_{32}.(-1) \quad \therefore -(1/2) = l_{32}$

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \qquad [U]=\begin{bmatrix} -2 & -3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}$$

First use the forward-substitution method to solve **LY = b**

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

- $y_1 = 1$
- $-y_1 + y_2 = 2 \rightarrow -1 + y_2 = 2 \quad \therefore y_2 = 3$
- $-\frac{1}{2} y_1 - \frac{1}{2} y_2 + y_3 = -3 \rightarrow -\frac{1}{2}(1) - \frac{1}{2}(3) + y_3 = -3 \quad \therefore y_3 = -1$

$$\therefore Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Then use back substitution to solve **UX = Y**

$$\begin{bmatrix} -2 & -3 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

- $\frac{3}{2} x_3 = -1 \rightarrow \quad \therefore x_3 = -\frac{2}{3}$
- $-x_2 + x_3 = 3 \rightarrow -x_2 - \frac{2}{3} = 3 \quad \therefore x_2 = -\frac{11}{3}$
- $-2x_1 - 3x_2 + 4x_3 = 1 \rightarrow -2x_1 - 3(-\frac{11}{3}) + 4(-\frac{2}{3}) = 1 \quad \therefore x_1 = \frac{11}{3}$

LU decomposition version of Gauss elimination

Gauss elimination can be use to decompose **[A]** into **[L]** and **[U]**. This can be easily seen for **[U]**, which is a direct product of the forward elimination.

Recall that the forward-elimination step is intended to reduced the original coefficient matrix **[A]** to the form

$$[\mathbf{U}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

which is in the desired upper triangular format.

The matrix [**L**] is also produced during the step where

$$l_{21} = \frac{a_{21}}{a_{11}} \quad , \quad l_{31} = \frac{a_{31}}{a_{11}} \quad , \quad l_{32} = \frac{a'_{32}}{a'_{22}}$$

$$\therefore \quad [\mathbf{L}] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

Hence, for example before:

Find matrix L and U using Gauss Elimination

$$-2x_1 - 3x_2 + 4x_3 = 1$$
$$2x_1 + 2x_2 - 3x_3 = 2$$
$$x_1 + 2x_2 - x_3 = -3$$

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$$

Example:

Use the Gauss-Seidel method to obtain the solution of the linear system

$$
\begin{aligned}
12x_1 + 3x_2 - x_3 &= 15 \\
x_1 + 8x_2 + x_3 &= 20 \\
2x_1 - x_2 + 10x_3 &= 30
\end{aligned}
$$

accurate to within $\varepsilon = 0.001$.

$$x_1^{(k+1)} = \frac{15 - 3x_2^{(k)} + x_3^{(k)}}{12}$$

$$x_2^{(k+1)} = \frac{20 - x_1^{(k+1)} - x_3^{(k)}}{8}$$

$$x_3^{(k+1)} = \frac{30 - 2x_1^{(k+1)} + x_2^{(k+1)}}{10}$$

Assume $\mathbf{x}^{(0)} = \mathbf{0}$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\Vert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\Vert_\infty$
0	0	0	0	
1	1.25	2.3438	2.9844	2.9844
2	0.9128	2.0129	3.0187	0.3372
3	0.9983	1.9979	3.0001	0.0855
4	1.0005	1.9999	2.9999	0.0022
5	1.0000	2.0000	3.0000	0.0005

$\therefore$ $\mathbf{X} = (x_1, x_2, x_3)^T = (1.0000, 2.0000, 3.0000)$