CHAPTER 4 INTEGRATION

4.1 Antiderivatives and Indefinite Integrals (page 244)

Function	Derivative
$F(x)$	$f(x)$

Definition 4.1 (Antiderivatives) ***(page 244)***

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f.

If $f(x) = 3x^2$, then $F(x) = x^3$

Therefore, $F(x) = x^3$ is an antiderivative of $f(x) = 3x^2$.

However, $G(x) = x^3 + 1$

$H(x) = x^3 - 10$

$I(x) = x^3 + \sqrt{2}$ etc,

are also an antiderivative of $f(x) = 3x^2$.

In general, the expression $F(x) + C$, where C an arbitrary constant, is an antiderivative of $f(x)$.

Definition 4.2 (Indefinite Integral) ***(page 245)***

A set of all antiderivatives of function $f(x)$ is called **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx$$

The symbol $\int$ is the **integral symbol**.

The function f is the **integrand** of the integral, and x is the **variable of integration**.

(page 246)

We can conclude that the differentiation expression

$$\frac{d}{dx}\left[f(x)\right] = f'(x)$$

is equivalent to the integral statement

$$\int f'(x)\,dx = f(x) + C$$

or $$\int \frac{d}{dx}\left[f(x)\right] dx = f(x) + C$$

Example (Table 4.1)

$$\frac{d}{dx}\left[x^2\right] = 2x \qquad \int 2x\,dx = x^2 + C$$

$$\frac{d}{dx}\left[\sin x\right] = \cos x \qquad \int \cos x\,dx = \sin x + C$$

Example 4.1 (page 246):

$$\frac{d}{dx}\left[-\frac{1}{4}\sqrt{1-4x^2}\right] = \frac{x}{\sqrt{1-4x^2}} \qquad \int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{4}\sqrt{1-4x^2} + C$$

4.2 Algebra of Indefinite Integrals (page 250)

Theorem 4.2 (Basic Properties of Indefinite Integrals)

(a) The constant factor k can be taken out from an integral, that is

$$\int kf(x)\,dx = k\int f(x)\,dx$$

(b) The integral of a sum or difference equals the sum or difference of the integrals, that is

$$\int \left[f(x) \pm g(x)\right] dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example 4.4 (page 250):

(a) $\int \left(3x^2 + 8\right) dx$

Remark (page 252):

We can combine the multiple constants of separate integrals into a single constant of integration, C.

Example (extra):

$$\int \frac{t^2 - 2t^4}{t^4}\,dt$$

4.3 The Fundamental Theorem of Calculus (page 254)

Indefinite integrals = integrals without the limits of integration

Definite integrals = integrals **with** the limits of integration

Theorem 4.3 (Fundamental Theorem of Calculus)

If a function $f(x)$ is continuous on the interval [a, b], then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is a function that $F'(x) = f(x)$ for all $x \in$ [a, b]

a = lower limit of integration

b = upper limit of integration

Theorem 4.4 (Basic Properties of Definite Integrals) 256

If $f(x)$ and $g(x)$ are continuous functions on the interval [a, b], then

(a) $\int_a^a f(x)\,dx = 0$, if $f(a)$ exists.

(b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

(c) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

(d) $\int_a^b k\,dx = k(b-a)$.

(e) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a \le c \le b$

(e) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example 4.7 (page 254):

(a) $\int_0^2 \left(3x^2 + 7\right) dx$

Example 4.8 (page 255):

If $y=(x+3)\sqrt{2x-3}$, show that $\dfrac{dy}{dx}=\dfrac{3x}{\sqrt{2x-3}}$. Hence, evaluate

$$\int_2^6 \frac{x}{\sqrt{2x-3}}\,dx.$$

Example 4.9 (Page 256):

(a) $\int_4^4 (2x+3)\,dx$ (b) $\int_4^0 (x-3)\,dx$

4.4 Techniques of Integration (page 259)

In the previous section, we learn how to integrate elementary functions using integral formulas. However, many integration problems are not expressed in such standard forms. Hence, we need some methods to convert the given integrals to elementary forms before carrying out the integrations.

There are some techniques of integration:

(a) Integration by substitutions
(b) Integration by parts
(c) Integration by tabular method
(d) Integration using partial fractions

4.4.1 Integration by Substitution

The purpose of this method is to *change the integrand into an expression of basic integral form*.

Step 1: Make appropriate choice of u, let $u = g(x)$.

Step 2: Obtain $\frac{du}{dx} = g'(x)$

Step 3: Substitute $u = g(x)$, $du = g'(x)dx$.

After this stage, the whole integral must be in terms of u. This means no more term in x can remain. If this step fails, we need to make another appropriate choice of u.

Step 4: Evaluate the integral obtained in terms u.

Step 5: Substitute u with $g(x)$, so that the final answer will be in terms of x.

Example 4.11 (page 261):

Evaluate the integrals by using the substitution $u = 4x + 1$.

(a) $\int (4x+1)\, dx$ (b) $\int (4x+1)^2\, dx$

In general, it can be shown by substitution $u = ax + b$ that

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C, \quad n \neq -1, \quad a \neq 0$$ (pg 262)

$$\int \frac{c}{ax+b}\, dx = \frac{c \ln|ax+b|}{a} + C$$

$$\int e^{ax+b}\, dx = \frac{e^{ax+b}}{a} + C$$

$$\int \cos(ax+b)\, dx = \frac{\sin(ax+b)}{a} + C$$

Example 4.12 (page 262):

(c) $\int e^{-3x+5}\ dx$ (d) $\int (10x-9)^{5/2}\ dx$

Example 4.13 (page 263):

(a) $\int \sin 15x\ dx$

Example 4.14 (page 264):

(b) $\int \frac{x}{x^2-4}\ dx$

Example 4.15 (page 265):

(a) $\int \frac{dx}{x-3}$

Example 4.16 (page 266):

(a) $\int \tan x\ dx$

Example 4.18 (page 268):

(a) $\int \frac{\ln x}{x}\ dx$

Example 4.20 (page 269):

(a) $\int 2x(x+2)^5\ dx$

Exercise at home: (Tutorial 7)

(Page 350) Exercise 4: Do no. 1, 9(b), 9(e), 9(h), 10(d), 10(e), 10(f), 11, 12.

4.4.2 Integration By Parts (page 275)

This method is important when the integrands involve products of algebraic and transcendental functions,

examples $\int x \ln x \, dx$, $\int x e^x \, dx$, $\int e^x \cos x \, dx$

Formula $\boxed{\int u \, dv = uv - \int v \, du}$ (page 276)

Example 4.26 (page 276):

(a) $\int x \ln x \, dx$

Remarks (page 277):

(i) when calculating the first integration, the constant k can be omitted

(ii) if the choice of u and dv causes the integrals to become more complicated, then the functions choice should be interchanged.

The following table lists several general integrals with suggestions for the choice of u and dv.

1. $\int x^n \ln x \, dx$ use substitution $u = \ln x$, $dv = x^n dx$

2. $\int e^{ax} \sin bx \, dx$ or $\int e^{ax} \cos bx \, dx$

   use substitution $u = \sin bx$ (or $\cos bx$), $dv = e^{ax} dx$

   or use $u = e^{ax}$, $dv = \sin bx \, dx$ (or $dv = \cos bx \, dx$),

3. $\int x^n e^{ax} \, dx$, $\int x^n \sin ax \, dx$ or $\int x^n \cos ax \, dx$

use substitution $u = x^n$, $dv = e^{ax} dx$, $\sin ax \, dx$ or $\cos ax \, dx$.

Example 4.27 (page 277):

(b) $\int_0^1 xe^{-x}\,dx$

** Some integrals require the repeated use of integration by parts formula.

Example 4.29 (page 279):

$\int x^2 \cos x\,dx$

Example 4.30 (page 280)

$\int e^x \cos 2x\,dx$

4.4.3 Tabular Method (page 285) (omitted)

Special case of integration by parts, which is suitable for the integrals that need repeated integrations and differentiations

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4.4.4 Integration by Partial Fractions (page 288)

A fraction of polynomials can be rewritten as a sum of simpler fractions that we may be able to integrate the resulting terms using basic integration formulas. This method is called the method of partial fractions.

Example 4.35 (page 289):

(a) $\int \frac{3x+2}{x^2+3x+2}\,dx$

(c) $\int \frac{6x+7}{(x+2)^2}\,dx$

Example 4.36 (page 290):

(a) $\int \frac{x}{x+1}\,dx$

(page 291 below)

Some time we need to use a substitution first to transform the given integral into the form suitable to the use of partial fractions. Only then the expression can be integrated.

Example 4.37 (page 292):

(a) $\int \frac{dx}{1-e^x}$, $\quad z = 1 - e^x$

Exercise at home: (Tutorial 8)

(page 352 & 354) Exercise 4: no. 19(a), 19(d), 19(h), 25(a), 25(d), 26(d), 27(c)

Extra question: $\int \frac{dx}{x^2-4}$

4.5 Integration of Trigonometric Functions (page 295)

4.5.1 Odd Powers of sin x and cos x (page 295)

Use identity $\sin^2 x + \cos^2 x = 1$

Example 4.38 (page 296)

(a) $\int \cos^3 x \, dx$ (b) $\int \cos^5 x \, dx$

Remarks (page 298):

To solve integrals of the form $\int \sin^m x \cos^n x \, dx$

where m **or** n **is an odd number**

(i) if m is odd, substitution: $u = \cos x$

identity: $\sin^2 x = 1 - \cos^2 x$

(ii) if n is odd, substitution: $u = \sin x$

identity: $\cos^2 x = 1 - \sin^2 x$

Example 4.39 (page 298)

(a) $\int \cos^3 x \sin x \, dx$

4.5.2 Even Powers of sin x and cos x (page 301)

Use identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ & $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Example 4.42 (page 302)

(a) $\int \sin^4 x \, dx$

Remarks (page 303):

To solve integrals of the form $\int \sin^m x \cos^n x \, dx$, where ***m* and *n* are even numbers**, use identities above repeatedly.

Example 4.42 (page 302)

(a) $\int \sin^4 x \cos^2 x \, dx$

4.5.3 – 4.5.9 (omitted)

4.5.10 Poducts of sin ax cos *bx*, sin *ax* sin *bx* and cos *ax* cos *bx* (page 319)

Use identities

$$\cos ax \sin bx = \frac{1}{2}[\sin(a+b)x - \sin(a-b)x]$$

$$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$

$$\sin ax \sin bx = -\frac{1}{2}[\cos(a+b)x - \cos(a-b)x]$$

$$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$$

Example 4.52 (page 319)

(a) $\int \sin 6x \cos 2x \, dx$

4.5.11 Rational Functions of sin *x* and cos *x* (pg 320)

The integrand is of the form

$$\frac{1}{a \sin x + b \cos x + c}$$

use substitution $t = \tan \frac{1}{2} x$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad \tan x = \frac{2t}{1-t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

These substitution can be remembered easily with the aid if

the triangle. $\tan \frac{1}{2} x = \frac{\sin x}{1+\cos x}$

Example 4.53 (page 320)

(a) $\int \frac{dx}{\cos x - \sin x - 1}$

Formula: $\int \sec x \, dx = \ln\left|\sec x + \tan x\right| + C$ (page 322)

$\int \operatorname{cosec} x \, dx = \ln\left|\operatorname{cosec} x - \cot x\right| + C$ (page 323)

4.6 Integration of Hyperbolic Functions (page 331) (omitted)

4.7 Integration of Irrational Functions (page 337)

4.7.1 Integrals involving $(ax+b)^{\frac{1}{2}}$

Integrate algebraic expression which contains only one irrational expression of the form $\sqrt{ax+b}$

Use substitution $z^2 = ax + b$

Example 4.62 (page 337)

(a) $\int x\sqrt{x+1}\, dx$

4.7.2 Integrals involving $\left(Ax^2+Bx+C\right)^{\frac{1}{2}}$

Integrate expressions containing $\sqrt{Ax^2+Bx+C}$.

By completing the square,

$\sqrt{(ax+b)^2+k^2}$, $\sqrt{(ax+b)^2-k^2}$, $\sqrt{k^2-(ax+b)^2}$

Or the expressions of the simply forms

$\sqrt{x^2+k^2}$, $\sqrt{x^2-k^2}$, $\sqrt{k^2-x^2}$

Expression	**Substitution**	
$\sqrt{x^2+k^2}$	$x=k\tan\theta$	(page 339)
$\sqrt{x^2-k^2}$	$x=k\sec\theta$	
$\sqrt{k^2-x^2}$	$x=k\sin\theta$	
$\sqrt{(ax+b)^2+k^2}$	$ax+b=k\tan\theta$	(page 342)
$\sqrt{(ax+b)^2-k^2}$	$ax+b=k\sec\theta$	
$\sqrt{k^2-(ax+b)^2}$	$ax+b=k\sin\theta$	

Example 4.64 (page 339)

(b) $\int\sqrt{5-x^2}\ dx$

Example 4.67 (page 343)

(a) $\int\frac{dx}{\sqrt{x^2+x-6}}$

Exercise at home: (Tutorial 9):

(page 326) Quiz 4E: no. 1b, 1c, 2f, 2m, 14a, 14d, 19a, 19b.

(page 348) Quiz 4G: no. 2a, 2d, 2e.