CHAPTER FOUR

PARTIAL DIFFERENTIAL EQUATIONS (PDE)

After completing these tutorials, students should be able to:

- determine whether the given partial differential equations are separable
- separate the PDE into appropriate ODE and solve the PDE by using the method of separation of variables
- solve the given initial value problem of heat equation
- separate the PDE into appropriate ODE and solve the PDE by using the method of separation of variables
- solve the given initial value problem of wave equation

Question 1

Determine if the following PDE's are separable. If so, separate the PDEs into appropriate ODEs:

(a) $$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)$$

Solution:

$$Let\ u(x,t) = \psi(x)G(t)$$

$$\psi(x)G'(t) = k\psi''(x)G(t) + \psi(x)G(t)$$
$$= G(t)[k\psi''(x) + \psi(x)]$$
$$\frac{G'(t)}{G(t)} = \frac{k\psi''(x) + \psi(x)}{\psi(x)}$$

(Separable)

(b) $$m\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)$$

Solution:

$$Let\ u(x,t) = \psi(x)G(t)$$

$$m\mathrm{y}(x)G'(t) = k\mathrm{y}''(x)G(t) + \mathrm{y}(x)G(t)$$
$$= G(t)[k\mathrm{y}''(x) + \mathrm{y}(x)]$$

$$\frac{G'(t)}{G(t)} = \frac{k\mathrm{y}''(x) + \mathrm{y}(x)}{\mathrm{y}(x)}$$

(Separable)

(c) $$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[k(x)\frac{\partial^2 u(x,t)}{\partial x}\right]$$

Solution:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}[k(x)\frac{\partial u(x,t)}{\partial x}] + u(x,t)$$
$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u(x,t)\partial k(x)}{\partial x} + k\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)$$

$$Let\ u(x,t) = \psi(x)G(t)$$

$$\psi(x)G^{'}(t) = \psi^{'}(x)G(t)k^{'}(x) + k(x)\psi''(x)G(t)) + \psi(x)G(t)$$
$$= G(t)[\psi^{'}(x)k^{'}(x) + k(x)\psi^{''}(x) + \psi(x)]$$
$$\frac{G^{'}(t)}{G(t)} = \frac{k(x)\psi^{''}(x) + \psi^{'}(x)k^{'}(x) + \psi(x)}{\psi(x)}$$ (Separable)

(d) $$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)$$

Solution:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)$$

$$Let\ u(x,t) = \psi(x)G(t)$$
$$\psi(x)G^{''}(t) = c^2\psi''(x)G(t) + \psi(x)G(t)$$
$$= G(t)[c^2\psi^{''}(x) + \psi(x)]$$
$$\frac{G^{''}(t)}{G(t)} = \frac{c^2\psi^{''}(x) + \psi(x)}{\psi(x)}$$ (Separable)

(e) $$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial u(x,t)}{\partial x} + u(x,t)$$

Solution:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - m\frac{\partial u(x,t)}{\partial t} + u(x,t)$$

$$Let\ u(x,t) = \psi(x)G(t)$$
$$\psi(x)G^{''}(t) = c^2\psi''(x)G(t) - m\psi^{'}(x)G(t) + \psi(x)G(t)$$
$$= G(t)[c^2\psi^{''}(x) - m\psi^{'}(x) + \psi(x)]$$
$$\frac{G^{'}(t)}{G(t)} = \frac{c^2\psi^{''}(x) - m\psi^{'}(x) + \psi(x)}{\psi(x)}$$ (Separable)

(f) $$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[c(x)\frac{\partial u(x,t)}{\partial x^2}\right] - \frac{\partial u(x,t)}{\partial x}$$

Solution:

$$\rho(x)\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left[c(x)\frac{\partial u(x,t)}{\partial x^2}\right] - \frac{\partial u(x,t)}{\partial x}$$

$Let\ u(x,t) = \psi(x)G(t)$

$$\rho(x)\psi(x)G''(t) = c'(x)\psi'(x)G(t) + c(x)\psi''(x)G(t)) - \psi'(x)G(t)$$
$$= G(t)[c'(x)\psi'(x) + c(x)\psi''(x) - \psi'(x)]$$

$$\frac{G''(t)}{G(t)} = \frac{c'(x)\psi'(x) + c(x)\psi''(x) - \psi'(x)}{\rho(x)\psi(x)}$$

(Separable)

Question 2

Consider the PDE:

$\frac{\partial u(x,t)}{\partial t} = 4\frac{\partial^2 u(x,t)}{\partial x^2}$ for each set of BCs and ICs and solve the initial value problem.

(a) $BCs: u(0,t) = 0; \frac{\partial u(\pi,t)}{\partial x} = 0$ and $IC: u(x,0) = x$

Solution:

$$\frac{\partial u(x,t)}{\partial t} = 4\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$G'(t)\psi(x) = 4G(t)\psi''(x)$$

$$\frac{G'(t)}{4G(t)} = \frac{\psi''(x)}{\psi(x)}$$

First solve the spatial equation to determine the valid value of λ

Case 1: $\lambda^2 > 0$

$\psi''(x) = \lambda^2\psi(x)$

$m^2 = \lambda^2$

$m = \pm\lambda$

$\therefore \psi(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

BC: $\psi(0) = C_1 + C_2$ (1)

BC: $\psi(2\pi) = C_1 e^{2\pi x} + C_2 e^{-2\pi x} = 0$ (2)

Compare (1) and (2) gives $C_1 = C_2 = 0$

No eigenvalue for $\lambda^2 > 0$

Case 2: $\lambda^2 = 0$

$\psi''(x) = 0$

$\psi'(x) = C_3$

$\psi(x) = C_3 x + C_4$

BCs: $\psi(0) = C_4 = 0$

$\psi(2\pi) = 2\pi C_3 = 0$

$C_3 = 0$

No eigenvalue for $\lambda^2 = 0$

Case 3: $\lambda^2 < 0$

$\psi''(x) = -\lambda^2 \psi(x)$

$m^2 = -\lambda^2$

$m = \pm \lambda i$

$\therefore \psi(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$

BCs: $\therefore \psi(0) = C_5 = 0$

$\psi(2\pi) = C_6 \sin 2\pi\lambda = 0$

$\sin 2\pi\lambda = 0, \pi, 2\pi, 3\pi, ...$

Eigenvalue, $\lambda_n = \dfrac{n}{2} \quad n = 0,1,2,3,.......$

Eigenfunction, $\psi_n(x) = C_n \sin \dfrac{n}{2} x$

For time equations $\lambda^2 < 0$

$G'(t) = -4\lambda^2 G(t)$

$G_n(t) = D_n e^{-4\lambda_n^2 t}$

There are an infinite number of separated solutions of Q2 (a), one for each n. There are

$u_n(x,t) = G_n(t)\psi_n(x)$

$$= b_n e^{-4\lambda_n^2 t} \sin \frac{(2n-1)}{2} x \qquad ; n = 1,2,3,4,......and\ b_n = D_n C_n$$

The sum of the solutions is again a solution, so

$$u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-4\lambda_n^2 t} \sin \frac{2n-1}{2} x$$

Determine b_n using initial condition and apply orthogonality

$$u_n(x,0) = x^2 - 2\pi x = \sum_{n=1}^{\infty} b_n \sin \frac{2n-1}{2} x$$

Note that $L = \pi$

$$b_n = \frac{2}{\pi} \int_0^{\pi} (x^2 - 2\pi x) \sin \frac{2n-1}{2} x\, dx$$

$$= \frac{2}{\pi} \left[\left[-\frac{2}{2n-1} (x^2 - 2\pi x) \cos \frac{2n-1}{2} \right]_0^{\pi} - \int_0^{\pi} \frac{2}{2n-1} (2x - 2\pi) \cos \frac{2n-1}{2} x\, dx \right.$$

$$= \frac{2}{\pi}\Big[\ \Big[\ -\frac{2}{2n-1}(x^2-2\pi x)\cos\frac{2n-1}{2}\ \Big]_0^{\pi} - \int_0^{\pi}\ \frac{2}{2n-1}(2x-2\pi)\cos\frac{2n-1}{2}x\,dx\ \Big]$$

$$= \frac{8}{\pi(2n-1)}\int_0^{\pi}\ (x-\pi)\cos\frac{2n-1}{2}x\,dx\ \Big]$$

$$= \frac{8}{\pi(2n-1)}\Big\{\ \Big[\ \frac{2}{2n-1}(x-\pi)\sin\frac{2n-1}{2}x\ \Big]_0^{\pi} - \int_0^{\pi}\ \frac{2}{2n-1}\sin\frac{2n-1}{2}x\,dx\ \Big\}$$

$$= \frac{16}{\pi(2n-1)^2}\Big\{\ \int_0^{\pi}\ \sin\frac{2n-1}{2}x\,dx\ \Big\}$$

$$= \frac{16}{\pi(2n-1)^2}\Big[\ \frac{2}{2n-1}\cos\frac{2n-1}{2}x\ \Big]_0^{\pi}\ \Big]$$

$$= \frac{32}{\pi(2n-1)^3}\big[\ 0-1\ \big]$$

$$= -\frac{32}{\pi(2n-1)^3}$$

$$u_n(x,t) = \sum_{n=1}^{\infty}\ -\frac{32}{\pi(2n-1)^3}e^{-4\lambda_n^2 t}\sin\frac{2n-1}{2}x$$

(b) $BCs: u(0,t)=0; u(2,t)=0$ and $IC: u(x,0)=\sin x$

Solution:

$$\frac{\partial u(x,t)}{\partial t} = 4\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$G'(t)\psi(x) = 4G(t)\psi''(x)$$

$$\frac{G'(t)}{4G(t)} = \frac{\psi''(x)}{\psi(x)}$$

First solve the spatial equation to determine the valid value of λ

Case 1: $\lambda^2 > 0$

$\psi''(x) = \lambda^2\psi(x)$

$m^2 = \lambda^2$

$m = \pm\lambda$

$\therefore \psi(x) = C_1e^{\lambda x} + C_2e^{-\lambda x}$

BC: $\psi(0) = C_1 + C_2$ (1)

BC: $\psi(2\pi) = C_1e^{2\pi x} + C_2e^{-2\pi x} = 0$ (2)

Compare (1) and (2) gives $C_1 = C_2 = 0$

No eigenvalue for $\lambda^2 > 0$

Case 2: $\lambda^2 = 0$

$\psi''(x) = 0$

$\psi'(x) = C_3$

$\psi(x) = C_3 x + C_4$

BCs: $\psi(0) = C_4 = 0$

$\psi(2\pi) = 2\pi C_3 = 0$

$C_3 = 0$

No eigenvalue for $\lambda^2 = 0$

Case 3: $\lambda^2 < 0$

$\psi''(x) = -\lambda^2 \psi(x)$

$m^2 = -\lambda^2$

$m = \pm \lambda i$

$\therefore \psi(x) = C_5 \cos \lambda x + C_6 \sin \lambda x$

BCs: $\therefore \psi(0) = C_5 = 0$

$\psi(2\pi) = C_6 \sin 2\pi\lambda = 0$

$\sin 2\pi\lambda = 0, \pi, 2\pi, 3\pi, ...$

Eigenvalue, $\lambda_n = \dfrac{n}{2} \quad n = 0,1,2,3,.......$

Eigenfunction, $\psi_n(x) = C_n \sin \dfrac{n}{2} x$

For time equations $\lambda^2 < 0$

$G'(t) = -4\lambda^2 G(t)$

$G_n(t) = D_n e^{-4\lambda_n^2 t}$

There are an infinite number of separated solutions, one of each n. there are

$u_n(x,t) = G_n(t)\psi_n(x)$

$= b_n e^{-4\lambda_n^2 t} \sin \dfrac{n}{2} x \qquad ; n = 1,2,3,4,......and\ b_n = D_n C_n$

The sum of the solutions is again a solution, so

$$u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-4\lambda_n^2 t} \sin \frac{n}{2} x$$

Determine b_n using initial condition and apply orthogonality

$$u_n(x,0) = \sin x = \sum_{n=1}^{\infty} b_n \sin \frac{n}{2} x$$

Note that $L = 2\pi$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} \sin x \sin\frac{n}{2}x\, dx$$
$$= \frac{1}{\pi}\left\{ \left[-\frac{2}{n}\sin x\cos\frac{n}{2}x \right]_0^{2\pi} + \int_0^{2\pi} \frac{2}{n}\cos x\cos\frac{n}{2}x\, dx \right\}$$
$$= \frac{2}{\pi n}\int_0^{2\pi} \cos x\cos\frac{n}{2}x\, dx \ ]$$
$$= \frac{2}{\pi n}\left\{ \left[\frac{2}{n}\cos x\sin\frac{n}{2}x \right]_0^{2\pi} + \int_0^{2\pi} \frac{2}{n}\sin x\sin\frac{n}{2}x\, dx \right\}$$
$$= \frac{4}{\pi n^2}\left\{ \int_0^{2\pi} \sin x\sin\frac{n}{2}x\, dx \right\}$$
$$= \frac{4}{n^2} b_n$$

Will give $b_n = 0$ except when $n = 2$

$$b_2 = \frac{2}{2\pi}\int_0^{2\pi} \sin x\sin x\, dx$$
$$= \frac{1}{\pi}\int_0^{2\pi} \sin^2 x\, dx$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (1-\cos^2 x)\, dx$$
$$= \frac{1}{2\pi}\left[\int_0^{2\pi} 1\, dx - \int_0^{2\pi} \cos 2x\, dx \right]$$
$$= \frac{1}{2\pi}\left\{ x\left[-\frac{1}{2}\sin 2x \right]_0^{2\pi} \right\}$$
$$= \frac{1}{2\pi}\left\{ 2\pi \right\}$$
$$= 1$$

From the argument above, the solution for u(x,t) exist when n=2, i.e.,

$\therefore u_n(x,t) = e^{-4t}\sin x$

Question 3

Consider the PDE:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

Solve the initial value problem subject to:

$$BCs:\begin{cases} u(-\pi,t) = u(\pi,t) \\ \dfrac{\partial u(-\pi,t)}{\partial x} = \dfrac{\partial u(\pi,t)}{\partial x} \end{cases} \quad \text{and} \quad IC: u(x,0) = x^2$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$G^{'}(t)\psi(x) = G(t)\psi^{"}(x)$$

$$\frac{G^{'}(t)}{G(t)} = \frac{\psi^{"}(x)}{\psi(x)}$$

First solve the spatial equation to determine the valid value of λ

Case 1: $\lambda^2 > 0$

$$\psi^{"}(x) = \lambda^2\psi(x)$$

$$m^2 = \lambda^2$$

$$m = \pm\lambda$$

$$\therefore \psi(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

BC: $u(-\pi,t) = u(\pi,t)$

$$\psi(-\pi) = \psi(\pi)$$

$$C_1 e^{\lambda x} + C_2 e^{-\lambda x} = C_1 e^{-\lambda x} + C_2 e^{\lambda x}$$

$$(C_1 - C_2)e^{-\lambda\pi} = (C_1 - C_2)e^{\lambda\pi} \qquad (1)$$

$$\psi^{'}(x) = C_1\lambda e^{\lambda x} - C_2\lambda e^{-\lambda x}$$

BC: $\dfrac{\partial u(-\pi,t)}{\partial t} = \dfrac{\partial u(x,t)}{\partial x}$

$$\psi^{'}(-\pi) = \psi^{'}(\pi)$$

$$C_1\lambda e^{\lambda x} - C_2\lambda e^{-\lambda x} = C_1\lambda e^{-\lambda x} - C_2\lambda e^{\lambda x}$$

$$(C_1 + C_2)e^{-\lambda\pi} = (C_1 + C_2)e^{\lambda\pi} \qquad (2)$$

Compare (1) and (2) gives $C_1 = C_2 = 0$

No eigenvalue for $\lambda^2 > 0$

Case 2: $\lambda^2 = 0$

$$\psi^{"}(x) = 0$$

$$\psi^{'}(x) = C_3$$

$$\psi(x) = C_3 x + C_4$$

BCs: $\psi(-\pi)=\psi(\pi)$

$$-C_3\pi+C_4=C_3\pi+C_4$$
$$C_3=0$$

No eigenvalue for $\lambda^2=0$

Case 3: $\lambda^2<0$

$$\psi''(x)=-\lambda^2\psi(x)$$
$$m^2=-\lambda^2$$
$$m=\pm\lambda i$$
$$\therefore \psi(x)=C_5\cos\lambda x+C_6\sin\lambda x$$

BC: $\therefore \psi(-\pi)=\psi(\pi)$

$$C_5\cos(-\lambda\pi)+C_6\sin(-\lambda\pi)=C_5\cos(\lambda\pi)+C_6\sin(\lambda\pi)$$
$$C_5\cos(\lambda\pi)-C_6\sin(\lambda\pi)=C_5\cos(\lambda\pi)+C_6\sin(\lambda\pi)$$
$$2C_6\sin(\lambda\pi)=0$$
$$\psi'(x)=-C_5\lambda\sin\lambda x+C_6\lambda\cos\lambda x$$

BC: $\dfrac{\partial\psi(-\pi)}{\partial x}=\dfrac{\partial\psi(\pi)}{\partial x}$

$$-C_5\lambda\sin(-\lambda\pi)+C_6\lambda\cos(-\lambda\pi)=-C_5\lambda\sin(\lambda\pi)+C_6\lambda\cos(\lambda\pi)$$
$$C_5\lambda\sin(\lambda\pi)+C_6\lambda\cos(\lambda\pi)=-C_5\lambda\sin(\lambda\pi)+C_6\lambda\cos(\lambda\pi)$$
$$2\lambda C_5\sin(\lambda\pi)=0$$
$$\sin(\lambda\pi)=0 \qquad \pi\lambda=0,\pi,2\pi,3\pi,......$$

Eigenvalue, $\lambda_n=n \quad n=0,1,2,3,.......$

Eigenfunction, $\psi_n(x)=C_n\cos nx+D_n\sin nx$

For time equations $\lambda^2<0$

$$G'(t)=-\lambda^2G(t)$$
$$G_n(t)=H_ne^{-\lambda_n^2t}=H_ne^{-n^2t}$$

There are an infinite number of separated solutions, one of each n. there are

$$u_n(x,t)=G_n(t)\psi_n(x)$$
$$=e^{-\lambda_n^2t}(A_n\cos nx+B_n\sin nx) \qquad ;n=1,2,3,4,......and\ A_n=H_nC_n, B_n=H_nD_n$$

The sum of the solutions is again a solution, so

$$u_n(x,t)=\sum_{n=1}^{\infty}e^{-\lambda_n^2t}(A_n\cos nx+B_n\sin nx)$$

Determine b_n using initial condition and apply orthogonality

$$u_n(x,0) = x^2 - x = \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx)$$

Note that $L = \pi$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 - x)\cos nx \, dx$$

$$= \frac{1}{\pi}\Big[\Big[\frac{1}{n}(x^2 - x)\sin nx \Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{n}(2x-1)\sin nx \, dx \Big]$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} (x-1)\sin nx \, dx \Big]$$

$$= -\frac{1}{n\pi}\Big\{ \Big[-\frac{1}{n}(2x-1)\cos nx \Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2}{n}\cos nx \, dx \Big\}$$

$$= -\frac{1}{n^2\pi}\Big[-\Big\{ (2x-1)\cos nx - (-2\pi - 1)\cos(-n\pi) \Big\} + \frac{2}{n}\sin nx \, dx \Big]_{-\pi}^{\pi}$$

$$= -\frac{1}{n^2\pi}\Big[-4\pi \Big]$$

$$= \frac{4(-1)^n}{n^2}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 - x)\sin nx \, dx$$

$$= \frac{1}{\pi}\Big[\Big[-\frac{1}{n}(x^2 - x)\cos nx \Big]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{n}(2x-1)\cos nx \, dx \Big]$$

$$= \frac{1}{n\pi}\Big\{ -\Big[(\pi^2 - \pi)\cos nx - (\pi^2 + \pi)\cos(-n\pi) \Big] + \int_{-\pi}^{\pi} (2x-1)\cos nx \, dx \Big\}$$

$$= -\frac{1}{n\pi}\Big[\Big[(-1)^n 2\pi + \frac{1}{n}(2x-1)\sin nx \Big]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2}{n}\sin nx \, dx \Big]$$

$$= -\frac{1}{n\pi}\Big[(-1)^n 2\pi + 0 - \int_{-\pi}^{\pi} \frac{2}{n}\sin nx \, dx \Big]$$

$$= -\frac{1}{n\pi}\Big[(-1)^n 2\pi + \frac{2}{n^2}\Big[\cos nx \Big]_{-\pi}^{\pi} \Big]$$

$$= -\frac{1}{n\pi}\Big[(-1)^n 2\pi + \frac{2}{n^2}(\cos nx - \cos(-n\pi)) \Big]$$

$$= \frac{2(-1)^n}{n}$$

$$u_n(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t}\Big(\frac{4(-1)^n}{n^2}\cos nx + \frac{2(-1)^n}{n}\sin nx\Big)$$

Question 4

Consider the PDE:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2}$$

For each set of BCs and ICs, solve the initial value problem.

(a)

$$BCs: \begin{cases} u(0,t) = 0 \\ u(p,t) = 0 \end{cases} \quad \text{and} \quad ICs: \begin{cases} u(x,0) = px - x^2 \\ \dfrac{\partial u(x,0)}{\partial t} = 0 \end{cases}$$

let $u = X(x)T(t)$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2}$$

$$XT'' = 9X''T$$

$$\frac{T''}{9T} = \frac{X''}{X} \text{ (separable)}$$

Case 1: $k = l^2$

$$\frac{X''}{X} = l^2$$

$$X'' - l^2 X = 0$$

$$m^2 - l^2 = 0$$

$$m^2 = l^2$$

$$m = \pm l$$

$$X = Ae^{lx} + Be^{-lx}$$

From boundary condition

$u(0,t) = 0$

$0 = Ae^{l(0)} + Be^{-l(0)}$

$0 = A + B$ ------------------ (1)

$u(p,t) = 0$

$0 = Ae^{l(p)} + Be^{-l(p)}$ -------------- (2)

By comparing (1) & (2), we get $A = B = 0$

$\therefore$ no eigenvalue for $k = l^2$

Case 2: $k = 0$

$$\frac{X''}{X} = 0$$
$$X'' = 0$$
$$m^2 = 0$$
$$m_1 = m_2 = 0$$
$$X = Ce^{(0)x} + Dxe^{(0)x}$$
$$X = C + Dx$$

From boundary conditions

$u(0,t) = 0$
$0 = C + D(0)$
$C = 0$

$u(\pi,t) = 0$
$0 = C + D(\pi)$
$D\pi = 0$
$D = 0$
$\therefore$ No eigenvalue for $k = 0$ because $C = D = 0$

Case 3: $k = -\lambda^2$

$$\frac{X''}{X} = -\lambda^2$$
$$X'' + \lambda^2 X = 0$$
$$m^2 + \lambda^2 = 0$$
$$m^2 = -\lambda^2$$
$$m = \pm\lambda i$$
$$X = E\cos\lambda x + F\sin\lambda x$$

From boundary conditions

$u(0,t) = 0$
$E = 0$

$u(\pi,t) = 0$
$E\cos\lambda\pi + F\sin\lambda\pi = 0$
Since $E = 0$, $F\sin\lambda\pi = 0$
$\sin\lambda\pi = 0$, because $F \neq 0$, otherwise trivial solution.

$\lambda\pi = 0, \pi, 2\pi, 3\pi, \ldots$
$\lambda = 0, 1, 2, 3, \ldots$
Eigenvalue, $\lambda_n = n$; n = 0, 1, 2, 3, . . .
$X = F\sin nx$

For time equation ($k = -\lambda^2$)

$$\frac{T''}{9T} = -\lambda^2$$
$$T'' = -9\lambda^2 T$$
$$m^2 = -9\lambda^2$$
$$m = \pm 3\lambda i$$

$$T = G\cos 3\lambda_n t + H\sin 3\lambda_n t$$
$$T = G\cos 3nt + H\sin 3nt$$

$$\therefore u_n = X(x)T(t)$$
$$= F\sin nt(G\cos 3nt + H\sin 3nt)$$
$$= \sin nt(A_n\cos 3nt + B_n\sin 3nt)$$
Where $A_n = FG$, $B_n = FH$

From initial boundary,
$$u(x,0) = \pi x - x^2 = \sum_{n=0}^{\infty} A_n \sin nx$$

L = π (from boundary conditions)
$$A_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\,dx$$
$$= 2\int_0^{\pi} x\sin nx\,dx - \frac{2}{\pi}\int_0^{\pi} x^2\sin nx\,dx$$
$$= 2\left[-\frac{\pi}{n}\cos nx + \frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$
$$= 2\left[-\frac{\pi}{n}\cos nx\right]_0^{\pi} - \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx + \frac{2}{n^3}\cos nx\right]_0^{\pi}$$
$$= \frac{4}{n^3\pi} - \frac{4}{n^3\pi}\cos n\pi$$
$$= \frac{4}{n^3\pi}(1 - \cos n\pi)$$
$$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$$

From initial boundary,
$$\frac{\partial u(x,0)}{\partial t} = 0$$
$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=0}^{\infty}\left[-3nA_n\sin nt + 3nB_n\cos 3nt\right]\sin nx$$
$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=0}^{\infty}\left[3nB_n\right]\sin nx = 0$$

$B_n = 0$

$\therefore u(x,t) = \sum_{n=0}^{\infty} \frac{4}{n^3\pi}\left[1-(-1)^n\right]\cos 3nt \sin 3nt$

(b) $BCs: \begin{cases} u(0,t) = 0 \\ \dfrac{\partial u(\frac{\pi}{2},t)}{\partial x} = 0 \end{cases}$ and $ICs: \begin{cases} u(x,0) = \sin x \\ \dfrac{\partial u(x,0)}{\partial x} = 0 \end{cases}$

let $u = X(x)T(t)$

$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2}$

$XT'' = 9X''T$

$\frac{T''}{9T} = \frac{X''}{X}$ (separable)

Case 1: $k = \lambda^2$

$\frac{X''}{X} = \lambda^2$

$X'' - \lambda^2 X = 0$

$m^2 - \lambda^2 = 0$

$m^2 = \lambda^2$

$m = \pm\lambda$

$X = Ae^{\lambda x} + Be^{-\lambda x}$

$X' = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$

From boundary condition

$u(0,t) = 0$

$0 = Ae^{\lambda(0)} + Be^{-\lambda(0)}$

$0 = A + B$ ------------------ (1)

$\frac{\partial u(\frac{\pi}{2},t)}{\partial x} = 0$

$0 = A\lambda e^{\lambda(\frac{\pi}{2})} - B\lambda e^{-\lambda(\frac{\pi}{2})}$ -------------- (2)

By comparing (1) & (2), we get A = B = 0

$\therefore$ No eigenvalue for $k = \lambda^2$

Case 2: $k = 0$

$\frac{X''}{X} = 0$

$X'' = 0$

$m^2 = 0$

$m_1 = m_2 = 0$
$X = Ce^{(0)x} + Dxe^{(0)x}$
$X = C + Dx$
$X^{'} = D$

From boundary conditions
$u(0,t) = 0$
$0 = C + D(0)$
$C = 0$

$$\frac{\partial u(\frac{\pi}{2},t)}{\partial x} = 0$$
$D = 0$
$\therefore$ No eigenvalue for $k = 0$ because $C = D = 0$

Case 3: $k = -\lambda^2$
$$\frac{X^{''}}{X} = -\lambda^2$$
$X^{''} + \lambda^2 X = 0$
$m^2 + \lambda^2 = 0$
$m^2 = -\lambda^2$
$m = \pm \lambda i$
$X = E\cos\lambda x + F\sin\lambda x$

From boundary conditions
$u(0,t) = 0$
$E = 0$

$$\frac{\partial u(\frac{\pi}{2},t)}{\partial x} = 0$$
$X^{'} = -\lambda E\sin\lambda x + \lambda F\cos\lambda x$
$X^{'} = \lambda F\cos\lambda x \quad (E = 0)$

$\lambda F\cos\lambda x = 0$

$\cos\lambda x = 0$ because $F \neq 0$ otherwise trivial solution.

$$\frac{\pi}{2}\lambda = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$= \frac{(2n-1)\pi}{2}; \ n = 1, 2, 3, \ldots$$

Eigenvalues, $\lambda_n = 2n-1; \ n = 1, 2, 3, \ldots$

$$X = F \sin(2n-1)x$$

For time equation ($k = -\lambda^2$)

$$\frac{T''}{9T} = -\lambda^2$$

$$T'' = -9\lambda^2 T$$

$$m^2 = -9\lambda^2$$

$$m = \pm 3\lambda i$$

$$T = G\cos 3\lambda_n t + H \sin 3\lambda_n t$$

$$T = G\cos 3(2n-1)t + H\sin 3(2n-1)t$$

$$\therefore u_n = X(x)T(t)$$

$$= F\sin(2n-1)t\left[G\cos 3(2n-1)t + H\sin 3(2n-1)t\right]$$

$$= \sin(2n-1)t\left[A_n \cos 3(2n-1)t + B_n \sin 3(2n-1)t\right]$$

Where $A_n = FG$, $B_n = FH$

$$u_n(x,t) = \sum_{n=1}^{\infty} \sin(2n-1)t\left[A_n \cos 3(2n-1)t + B_n \sin 3(2n-1)t\right]$$

From initial conditions,

$$u(x,0) = \sin x = \sum_{n=0}^{\infty} A_n \sin(2n-1)x$$

$L = \frac{\pi}{2}$ (from boundary conditions)

$$A_n = \frac{2}{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin x \sin(2n-1)x \ dx$$

The solution exists when $n = 1$

$$A_n = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin x \sin x \ dx$$

$$= \frac{4}{p}\int_0^{\frac{p}{2}} \frac{1}{2}(1-\cos 2x)\, dx$$

$$= \frac{2}{p}\int_0^{\frac{p}{2}} (1-\cos 2x)\, dx$$

$$= \frac{2}{p}\left[x - \frac{1}{2}\sin 2x \right]_0^{\frac{p}{2}}$$

$$= \frac{2}{p}\left(\frac{p}{2}\right)$$

$$= 1$$

$$\frac{\partial u(x,t)}{\partial t} = \sum_{n=1}^{\infty}\left[-3(2n-1)A_n \sin 3(2n-1)t + 3(2n-1)B_n \cos 3(2n-1)t\right]\sin(2n-1)x$$

From initial conditions,

$$\frac{\partial u(x,0)}{\partial t} = \sum_{n=1}^{\infty}\left[3(2n-1)B_n\right]\sin(2n-1)x$$

$$\frac{\partial u(x,0)}{\partial t} = 0$$

$$\sum_{n=1}^{\infty}\left[3(2n-1)B_n\right]\sin(2n-1)x = 0$$

$$B_n = 0$$

$$\therefore u(x,t) = \cos 3t \sin x$$

Question 5

Solve the initial-boundary value problem:

$$u_{tt}(x,t)=4u_{xx}(x,t)\ ,\quad 0\prec x\prec 2\ ,\quad 0\prec t\prec\infty$$
$$u(0,t)=u(2,t)\ ,\quad 0\le t\prec\infty$$
$$u(x,0)=\frac{1}{2}\sin\frac{\pi x}{2}\ ,\quad u_t(x,0)=-\sin\frac{\pi x}{2}\ ,\quad 0\le t\le 2$$

$$u(x,t)=4u(x,t)$$
$$XT''=4X''T$$
$$\frac{T''}{4T}=\frac{X''}{X}\qquad (separable)$$

Case 1

$$k=\lambda^2$$
$$\frac{X''}{X}=\lambda^2$$
$$X''=\lambda^2 X$$
$$m^2=\lambda^2$$
$$m=\pm\lambda$$
$$X=Ae^{\lambda x}+Be^{-\lambda x}$$

$From\ BCs:$

$$u(0,t)=0$$
$$0=Ae^{\lambda 0}+Be^{-\lambda 0}$$
$$A+B=0-----(1)$$
$$u(2,t)=0$$
$$0=Ae^{2\lambda}+Be^{-2\lambda}----(2)$$

Comparing (1) &(2), we get $A=B=0$

$\therefore$ no eigenvalue

Case 2

$$k=0$$
$$\frac{X''}{X}=0$$
$$X''=0$$
$$m^2=0$$
$$m=0\ @\ m=0$$
$$X=C+Dx$$

$$From\ BCs:$$
$$u(0,t)=0$$
$$0=C+D(0)$$
$$C=0$$
$$u(2,t)=0$$
$$0=C+D(2)$$
$$D=0$$

$\therefore$ no eigenvalue

Case3

$$k=-\lambda^2$$
$$\frac{X''}{X}=-\lambda^2$$
$$X''=-\lambda^2 X$$
$$m^2=-\lambda^2$$
$$m=\pm\lambda i$$
$$X=E\cos\lambda x+F\sin\lambda x$$

$$From\ BCs:$$
$$u(0,t)=0$$
$$0=E\cos 0+F\sin 0$$
$$E=0$$

$$u(2,t)=0$$
$$0=F\sin 2\lambda$$
$$F\neq 0,\quad otherwise\ trivial\ solution$$

$$\sin 2\lambda=0$$
$$2\lambda=\pi,2\pi,3\pi,4\pi,\ldots$$
$$\lambda=\frac{n\pi}{2};\ n=1,2,3,\ldots$$
$$X=F\sin\frac{n\pi}{2}x$$

For time equation ($k=-\lambda^2$)

$$\frac{T''}{4T}=-\lambda^2$$

$$T''=-4\lambda^2 T$$

$$m^2=-4\lambda^2$$

$$m=\pm 2\lambda i$$

$$T=G\cos 2\lambda t+H\sin 2\lambda t$$

$$T=G\cos n\pi t+H\sin n\pi t$$

$$U=XT$$

$$=\left(F\sin\frac{n\pi}{2}\lambda\right)\left(Gcon\pi x+H\sin n\pi x\right)$$

$$=(\mathrm{A_n}\cos n\pi t+B_n\sin n\pi t)\sin\frac{n\pi}{2}x$$

$$u_n(x,t)=\sum_{n=1}^{\infty}(\mathrm{A_n}\cos n\pi t+B_n\sin n\pi t)\sin\frac{n\pi}{2}x$$

$$u(x,0)=\frac{1}{2}\sin\frac{\pi x}{2}=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi}{2}$$

$L=4$ (from BCs)

$$A_n=\frac{2}{4}\int_0^4\frac{1}{2}\sin\frac{\pi}{2}x\sin\frac{n\pi}{2}x\,dx$$

$$=\frac{1}{4}\int_0^4\sin\frac{\pi}{2}x\sin\frac{n\pi}{2}x\,dx$$

Solution exists when $n=1$

$$A_1=\frac{2}{4}\int_0^4\frac{1}{2}\sin\frac{\pi}{2}x\sin\frac{\pi}{2}x\,dx$$

$$=\frac{1}{4}\int_0^4\frac{1}{2}(1-\cos\pi x)dx$$

$$=\frac{1}{8}\left[x-\frac{1}{\pi}\sin\pi x\right]_0^4$$

$$=\frac{1}{2}$$

$$u_t(x,t)=\sum_{n=1}^{\infty}\left[-n\pi A_n\sin n\pi t+n\pi B_n\cos n\pi t\right]\sin\frac{n\pi}{2}x$$

$$u_t(x,0)=-\sin\frac{\pi}{2}x=n\pi B_n\sin\frac{n\pi}{2}x$$

$$n\pi B_n = \frac{2}{4}\int_0^4 -\sin\frac{\pi}{2}x\sin\frac{n\pi}{2}\,dx$$
$$= -\frac{1}{2}\int_0^4 \sin\frac{\pi}{2}x\sin\frac{n\pi}{2}\,dx$$

Solution exists when n = 1

$$\pi B_1 = \frac{2}{4}\int_0^4 -\sin\frac{\pi}{2}x\sin\frac{\pi}{2}x\,dx$$
$$= -\frac{1}{4}\int_0^4 (1-\cos\pi x)\,dx$$
$$= -\frac{1}{4}\left[x-\frac{1}{\pi}\sin n\pi\right]_0^4$$
$$= -1$$
$$B_n = -\frac{1}{\pi}$$
$$\therefore u(x,t) = \frac{1}{2}\cos\pi t\sin\frac{\pi}{2}x - \frac{1}{\pi}\sin\pi t\sin\frac{\pi}{2}x$$

Question 6

Suppose we pluck a string by pulling it upward and release it from rest. We model the initial displacement in an idealized manner; using a triangular function defined on $0 \le x \le 1$. Since the string is released from rest, its initial velocity is zero. Therefore, the problem to be solved is:

$$\mathrm{u}_{tt}(x,t) = c^2 u_{xx}(x,t),\quad 0 \prec x \prec 1,\quad 0 \prec t \prec \infty$$
$$\mathrm{u}(0,t) = u(l,t) = 0,\quad 0 \le t \prec \infty$$
$$\mathrm{u}(x,0) = \begin{cases} \dfrac{2u_0 x}{l} & ,\ 0 \le x \le \dfrac{l}{2} \\ \dfrac{2u_0 x}{l}(l-x), & \dfrac{l}{2} \prec x \le l \end{cases}$$
$$\mathrm{u}_t(x,0) = 0,\quad 0 \le x \le l$$

The positive constant u_0 is the maximum initial string.

Let $u(x,t) = X(x)T(t)$

$$\mathrm{u}_{tt}(x,t) = c^2 u_{xx}(x,t)$$
$$XT'' = c^2 X''T$$
$$\frac{T''}{c^2 T} = \frac{X''}{X} \quad (separable)$$

Case 1 : $k = \lambda^2$

$$\frac{X^{"}}{X} = \lambda^2$$

$X^{"} = \lambda^2 X$

$m^2 = \lambda^2$

$m = \pm\lambda$

$\therefore X = Ae^{\lambda x} + Be^{-\lambda x}$

From BCs:

$u(0,t) = 0$

$0 = A + B$ ………………….(1)

$u(1,t) = 0$

$0 = Ae^{x} + Be^{-x}$ ……………..(2)

Comparing (1) and (2), $A = B = 0$

$\therefore$ No eigenvalue for $k = \lambda^2$

Case 2 : $k = 0$

$$\frac{X^{"}}{X} = 0$$

$X^{"} = 0$

$m^2 = 0$

$m_1 = m_2 = 0$

$\therefore X = Ce^{(0)x} + Dxe^{-(0)x}$

$= C + Dx$

From BCs:

$u(0,t) = 0$

$0 = C + D(0)$

$C = 0$

$u(1,t) = 0$

$0 = C + D(1)$

$0 = 0 + D(1)$

$D = 0$

$\therefore$ No eigenvalue for $k = \lambda^2$ because $C = D = 0$

Case 3 : $k=-\lambda^2$

$$\frac{X''}{X}=-\lambda^2$$

$$X''=-\lambda^2 X$$

$$m^2=-\lambda^2$$

$$m=\pm\lambda i$$

$$\therefore X=E\cos\lambda x+F\sin\lambda x$$

From BCs:

$$u(0,t)=0$$

$$0=E+F(0)$$

$$E=0$$

$$u(1,t)=0$$

$$0=E\cos\lambda+F\sin\lambda$$

$$0=(0)\cos\lambda+F\sin\lambda$$

$$F\sin\lambda=0$$

$F\neq 0$, otherwise trivial solution.

$$\sin\lambda=0$$

$$\lambda=\pi,2\pi,3\pi,\ldots$$

$$\lambda_n=n\pi;\;\; n=1,2,3,\ldots$$

Eigenvalues , $\lambda_n=n\pi$

$$X_n=F\sin n\pi x$$

For time equation, $k=-\lambda^2$

$$\frac{T''}{c^2T}=-\lambda^2$$

$$T''=-\lambda^2c^2T$$

$$m^2=-\lambda^2c^2$$

$$m=\pm\lambda ci$$

$$T_n=G\cos c\lambda_n t+H\sin c\lambda_n t$$

$$T_n=G\cos cn\pi t+H\sin cn\pi t$$

$$u_n=X_nT_n$$

$$=F\sin n\pi x\left[G\cos cn\pi t+H\sin cn\pi t\right]$$

$$=\sin n\pi x\left[A_n\cos cn\pi t+B_n\sin cn\pi t\right]$$

Where $A_n=FG$ and $B_n=FH$

$$u(x,t)=\sum_{n=1}^{\infty}\sin n\pi x\left[A_n\cos cn\pi t+B_n\sin cn\pi t\right]$$

From ICS :

$$u(x,0)=\begin{cases}\dfrac{2u_0x}{l} & ,\ 0\le x\le \dfrac{l}{2}\\[2mm] \dfrac{2u_0x}{l}(l-x)\ , & \dfrac{l}{2}\prec x\le l\end{cases}$$

$L=l$ (from boundary condition)

$$A_n=\frac{2}{l}\int_0^{\frac{l}{2}}\frac{2u_0x}{l}\sin n\pi x\,dx+\frac{2}{l}\int_{\frac{l}{2}}^{l}\frac{2u_0x}{l}(l-x)\sin n\pi x\,dx$$

$$=\frac{8u_0}{n^2\pi^2}\cos n\pi$$

$$=\frac{8u_0}{n^2\pi^2}(-1)^n$$

$$\therefore u(x,t)=\frac{8u_0}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}(-1)^n\cos cn\pi t\sin n\pi x$$