CHAPTER FIVE

COMPLEX NUMBERS

5.0 Introduction

In solving quadratic equations we notice that there are cases when we can obtain real different roots or repeated roots. However there are times when no real roots can be obtained. In this case we need complex numbers

Complex numbers are useful abstract quantities that can be used in calculations and result in physically meaningful solutions. Complex numbers are used in a number of fields, including: engineering, electromagnetism, quantum physics, applied mathematics, and chaos theory.

5.1 Imaginary numbers

Consider, $x^2 = -9$.

This equation has no real solution. To solve the equation, we will introduce an imaginary number;

Definition 5.1 (Imaginary Number)

The imaginary number i is defined as

$$i^2 = -1 \text{ or } i = \sqrt{-1}$$

Using the definition, we will get,

$$\begin{aligned} x^2 &= -9 \\ x &= \pm\sqrt{-9} \\ &= \pm\sqrt{9}\sqrt{-1} \\ &= \pm 3i \end{aligned}$$

5.2 Complex Numbers

If the real numbers and the imaginary numbers are combined with the operation of addition or subtraction, a number is formed, called the complex numbers.

For example: $3+4i, 7-5i, -1+3i, -1-4i$

Definition 5.2 (Complex Numbers)

If z is a ***complex number***, then it can be expressed in the form

$$z = x + iy,$$

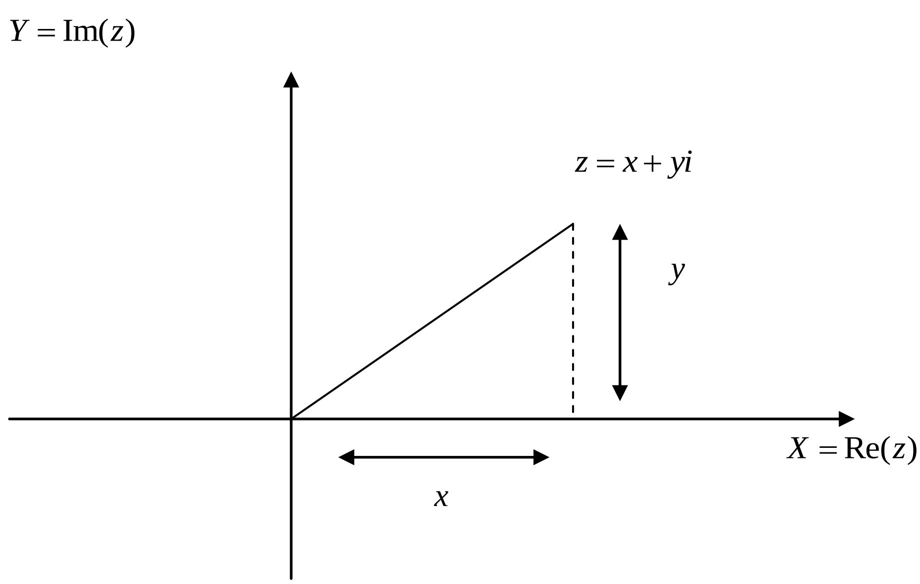
where $x, y \in R$ and $i = \sqrt{-1}$.

The ***real part*** of the complex number z is x.

The **imaginary part** of the complex number z is y. Or frequently represented as $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

5.2.1 Argand Diagram

We can graph complex numbers using an **Argand Diagram**.



5.2.2 Equality of 2 Complex Numbers

Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $z_1, z_2 \in C$.

Two complex numbers are equal if and only if the real parts and the imaginary parts are respectively equal.

So, if $z_1 = z_2$, then $x_1 = x_2$ and $y_1 = y_2$.

5.3 Algebraic Operations on Complex Numbers

5.3.1 Addition and subtraction

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

5.3.2 Multiplication

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, and k is a constant, then

(i) $$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1) \end{aligned}$$

(ii) $k\,z_1 = k\,x_1 + i\,ky_1$

5.3.3 Complex Conjugate

If $z = x + iy$ then the **conjugate** of z is denoted as $\overline{z} = x - iy$.

Note that $z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 + y^2 \in R$.

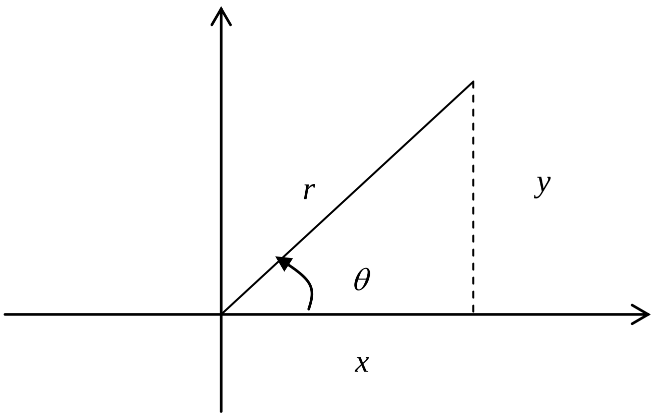
5.3.4 Division

If we are dividing the complex number, the denominator must be converted to a real number.

In order to do that, multiply both the denominator and numerator by complex conjugate of the denominator.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{(x_1 x_2 + y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)}{(x_2)^2 + (y_2)^2} \end{aligned}$$

5.4 Polar Form of Complex Numbers



- The distance *OP*, is known as the **modulus** of *z*, and is denoted as $|z|$.
  - $|z| = r = \sqrt{x^2 + y^2}$
- The **argument** of *z* (arg *z*), θ is the angle between the positive *x*-axis and the line *OP*.
  $\tan\theta = \dfrac{y}{x}$
- θ is known as the **principle argument** if $-\pi \le \theta \le \pi$.
- **From the diagram above, we can see that**
  $x = r\cos\theta$ and $y = r\sin\theta$.

  Then, *z* can be written as

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= r\,\text{cis}\theta \end{aligned}$$

5.4.1

Multiplication/ Division of complex Numbers In Polar Coordinates

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiplication: $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

5.5 Eulers's Formula

Definition 5.3

Euler's formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It follows that

$$e^{in\theta} = \cos n\theta + i\sin n\theta$$

Form the definition, if z is any complex number with modulus r and $\text{Arg}(z) = \theta$, then

$$\begin{aligned} z &= r\cos\theta + i\,r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{\,i\theta} \end{aligned}$$

5.5.1 Euler's Formula And The *n*-th Power Of A Complex Number

We know that a complex number can be express as $z = re^{\,i\theta}$, then

$$\begin{aligned} z^2 &= r^2 e^{\,i2\theta} \\ z^3 &= r^3 e^{\,i3\theta} \\ z^4 &= r^4 e^{\,i4\theta} \\ &\vdots \\ z^n &= r^n e^{\,in\theta} \end{aligned}$$

5.5.2 Euler's Formula And The *n*-th Roots Of A Complex Number

The *n*-th roots of a complex number can be find using the Euler's formula. Note that

$$z = re^{\,i(\theta + 2k\pi)}$$

Then,

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{\,i\left(\frac{\theta+2k\pi}{2}\right)}, \qquad \text{for} \qquad k = 0,1$$

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\,i\left(\frac{\theta+2k\pi}{3}\right)}, \qquad \text{for} \qquad k = 0,1,2$$

$$\vdots$$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2k\pi}{n}\right)}, \qquad \text{for} \qquad k = 0,1,2,\ldots,\ n-1$$

5.6 De Moivre's Theorem

Definition 5.8 (De Moivre's Theorem)

If $z = r\cos\theta + i\,r\sin\theta$ and $n \in R$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

De Moivre 's theorem also can be used to find all of the ***n*th roots** of any number.

If $z^n = r(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} z &= [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} \\ &= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right] \end{aligned}$$

for $k = 0,1,2,\ldots,\ n-1$.

Substituting $k = 0,1,2,\ldots,\ n-1$ yields the nth roots of the given complex number.