CHAPTER FIVE

FUNCTIONS AND GRAPHS

Introduction

In this chapter, the important of functions are discussed. We will learn how to find the range of a function and also how to sketch the graph of some functions. This topic will introduce the composite functions and how to calculate the composite functions. Finally, the inverse functions will be discussed based on the given functions.

Objectives

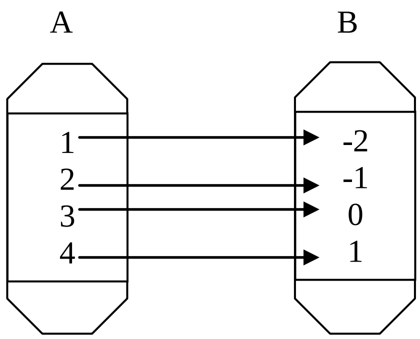
After completing these tutorials, students should be able to:

- draw the arrow diagram for the given function, f and find the range of f.
- Sketch the graph for the given function in order to find the range of the function.
- Find the domain for the given function so that it is defined.
- Define the composite function (*fog*), if the functions *f* and *g* are given.
- Define the function *g*, if the function *f* and the composite function (*fog*) are given.
- Find the inverse of the given function and state its domain.
- Sketch the graph of each of the given functions and its inverse.

Question 1

If $A = \{1, 2, 3, 4\}$ and B is a set of all integers, draw the arrow diagram showing the function $f : x \to x - 3$ where $x \in A$. What is the range of f .

Solution:



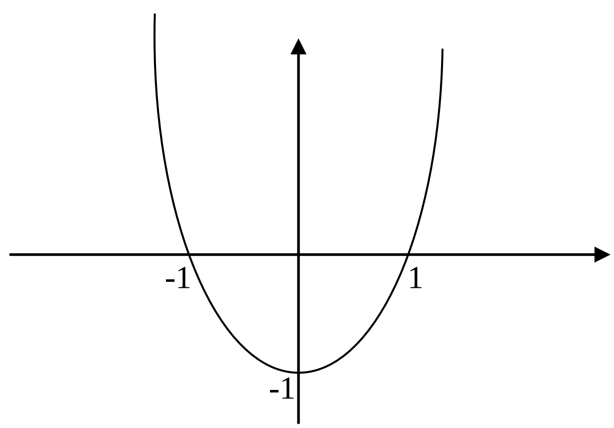
Range = $\{-2, -1, 0, 1\}$

Question 2

Sketch the graph $y = x^2 - 1$ for $-3 \le x \le 2$. Find its range.

Solution:

x = -3, -2, -1, 0, 1, 2
y = 8, 3, 0, -1, 0, 3
$\therefore range = [-1, 8]$



Question 3

Given that $f(x) = x^2 - 3x.$ *Evaluate* $f(1), f(2), f(-1).$

Solution:

$f(1) = (1)^2 - 3(1) = -2$

$f(2) = (2)^2 - 3(2) = -2$

$f(-1) = (-1)^2 - 3(-1) = 4$

Question 4

For what value of x is the function $f(x) = \frac{x}{(x-1)(x-3)}$ defined?

Solution:

All values of $x \in R$ except x = 1 and x = 3.

Question 5

For what value of x is the function $f(x) = (x-5)(x-7)$ negative?. For what value of x is the function $g(x) = \sqrt{(x-5)(x-7)}$ defined?

Solution:

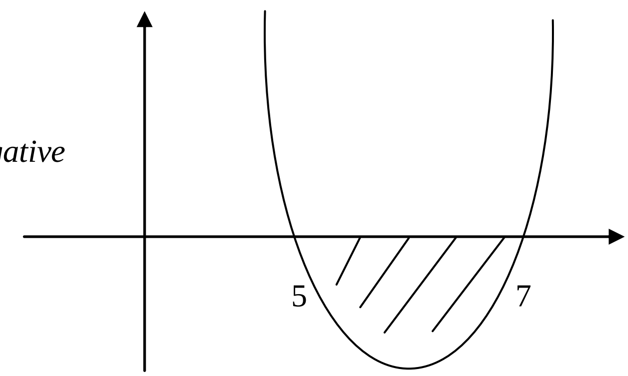
$f(x) = (x-5)(x-7)$

$= x^2 - 12x + 35$

$\therefore 5 \langle x \langle 7 \Rightarrow the\ \ function\ \ f(x)\ is\ negative$

$g(x) = \sqrt{(x-5)(x-7)}$

$\therefore x \leq 5\ \ or\ \ x \geq 7,\ g(x)\ is\ defined.$



Question 6

$If\ \ \ f(x) = 2x,\ g(x) = x-1,\ h(x) = x^2,\ find:$

(a) *(fog)(x)*

Solution:

(fog)(x) = f(g(x))
= f(x-1)
= 2(x-1)
= 2x – 2

(b) *(fogoh)(x)*

Solution

(fogoh)(x) = f(g(h(x)))
= f(g(x^2))
= f(x^2-1)
= 2 (x^2-1)
= $2x^2 - 2$

(c) $(f^2 \circ g)(x)$

Solution:

$(f^2 \circ g)(x) = f(f(g(x)))$
$= f(f(x-1)$
$= f(2(x-1))$
$= 4(x-1)$

(d) $(g \circ h^2)(2)$

Solution:

$(g \circ h^2)(x) = g(h(h(x)))$
$= g(h(x^2))$
$= g(x^4)$
$= x^4 - 1$

$(g \circ h^2)(2) = (2)^4 - 1 = 15$

(e) $(f \circ g)(x+1)$

Solution:

$(f \circ g)(x) = 2(x-1)$
$(f \circ g)(x+1) = 2((x+1)-1)$
$= 2x$

(f) $(g \circ g)(a-1)$

Solution:

$(g \circ g)(a-1) = (gog)(x) = g(g(x))$
$= g(x-1)$
$= (x-1) - 1$
$= x - 2$

$(g \circ g)(a-1) = (a-1) - 2$
$= a - 3$

Question 7

If $f(x) = 2x - 3$ *and* $(fog)(x) = 2x + 1$, *find* $g(x)$.

Solution:

$$(fog)(x) = f(g(x)) = 2x + 1$$
$$2(g(x)) - 3 = 2x + 1$$
$$2(g(x)) = 2x + 4$$
$$g(x) = \frac{2x + 4}{2}$$
$$= x + 2$$

Question 8

If $f(x) = x - 1$ *and* $(gof)(x) = 3 + 2x - x^2$, *find* $g(x)$.

Solution:

$$(gof)(x) = 3 + 2x - x^2$$
$$g(f(x)) = 3 + 2x - x^2$$
$$g(x - 1) = 3 + 2x - x^2$$
Assume that $y = x - 1 \Rightarrow x = y + 1$
$$g(y) = 3 + 2(y + 1) - (y + 1)^2$$
$$= 3 + 2y + 2 - y^2 - 2y - 1$$
$$= 5 - y^2 - 1$$
$$= 4 - y^2$$
$$\therefore g(x) = 4 - x^2$$

Question 9

The function f and the composite function (fog) are defined as follows:

$f : x \to x^2 + 1,\ x \geq 0$ *and* $(fog) : x \to x^2 - 2x + 2,\ x \geq 1$. *Define the function* g

Solution:

$(fog)(x) = f(g(x)) = x^2 - 2x + 2$. *Given that* $f(x) = x^2 + 1$
$$\therefore f(g(x)) = (g(x))^2 + 1 = x^2 - 2x + 2$$
$$(g(x))^2 = x^2 - 2x + 1$$
$$= (x - 1)^2$$
$$g(x) = x - 1$$

Question 10

The function g and the composite fucntion (gof) are defined as follows:

$g: x \to \ln x,\ x \rangle 0\ \ and (gof): x \to 2\ln(x=1),\ x \rangle -1.\ Define\ the\ \ function\ \ \ f$

Solution:

$(gof)(x) = g(f(x)) = 2\ln(x+1).\ Given\ that\ \ g(x) = \ln x$

$$\ln(f(x)) = 2\ln(x+1)$$

$$\ln(f(x)) = \ln(x+1)^2$$

$$f(x) = (x+1)^2$$

Question 11

Find the inverse of each of the following functions, stating its domain.

(a) $f(x) = x^2 + 1;\ \ x \geq 0$

Solution:

$f(x) = x^2 + 1;\ \ x \geq 0$

$y = x^2 + 1$

$x^2 = y - 1$

$x = \sqrt{y-1}$

$f^{-1}(y) = \sqrt{y-1}$

$f^{-1}(x) = \sqrt{x-1};\ \ x \geq 1$

(b) $f(x) = (x-1)^2;\ \ x \geq 1$

Solution:

$f(x) = (x-1)^2;\ \ x \geq 1$

$y = (x-1)^2$

$\sqrt{y} = (x-1)$

$x = \sqrt{y} + 1$

$f^{-1}(y) = \sqrt{y} + 1$

$f^{-1}(x) = \sqrt{x} + 1;\ x \geq 0$

(c) $g(x) = (x-2)(x-4); x \geq 3$

Solution:

$g(x) = (x-2)(x-4); x \geq 3$

$y = (x-2)(x-4)$

$y = x^2 - 6x + 8$

$y = (x-3)^2 - 1$

$y + 1 = (x-3)^2$

$x - 3 = \pm\sqrt{y+1}$

$x - 3 = \sqrt{y+1}$

$x = \sqrt{y+1} + 3$

$g^{-1}(y) = \sqrt{y+1} + 3$

$g^{-1}(x) = \sqrt{x+1} + 3; \;\; x \geq -1$

(d) $h(x) = \dfrac{2}{x-3}; x \neq 3$

Solution:

$h(x) = \dfrac{2}{x-3}$

$y = \dfrac{2}{x-3}$

$yx - 3y = 2$

$x = \dfrac{2+3y}{y}$

$\;\;= \dfrac{2}{y} + 3$

$h^{-1}(y) = \dfrac{2}{y} + 3$

$h^{-1}(x) = \dfrac{2}{x} + 3; \;\; x \neq 0$

(e) $h(x)=\frac{x+2}{x-2}; x\neq 2$

Solution:

$$h(x)=\frac{x+2}{x-2}; x\neq 2$$

$$y=\frac{x+2}{x-2}$$

$$yx-2y=x+2$$

$$yx-x=2+2y$$

$$x(y-1)=2+2y$$

$$x=\frac{2+2y}{y-1}$$

$$h^{-1}(y)=\frac{2+2y}{y-1}$$

$$h^{-1}(x)=\frac{2+2x}{x-1}, x\neq 1$$

Question 12

The function f and g are defined by:

$f: x\rightarrow 2(x+3)^2-5;\ \ x\leq -3$

$g: x\rightarrow \sqrt{\frac{x+5}{2}}; x\geq -5$

(a) Show that f is one-to-one function and find an expression for $f^{-1}(x)$.

Solution:

$$f(x)=2(x+3)^2-5; \qquad x\leq -3$$

$$=2(x^2+6x+9)-5$$

$$=2x^2+12x+18-5$$

$$=2x^2+12x+13$$

$$2(x_1+3)^2-5=2(x_2+3)^2-5$$

$$(x_1+3)^2=(x_2+3)^2$$

$$(x_1+3)=(x_2+3)$$

$$x_1=x_2$$

$$f(x)=2(x+3)^2-5$$
$$y=2(x+3)^2-5$$
$$2(x+3)^2=y+5$$
$$(x+3)^2=\frac{y+5}{2}$$
$$x+3=\pm\sqrt{\frac{y+5}{2}}$$
$$x=-\sqrt{\frac{y+5}{2}}-3 \qquad \left\{Given\ x\le -3 \Rightarrow x+3\le 0 \therefore Choose -\sqrt{\frac{y+5}{2}}\right\}$$
$$f^{-1}(y)=-\sqrt{\frac{y+5}{2}}-3$$
$$f^{-1}(x)=-\sqrt{\frac{x+5}{2}}-3 \quad , \quad x\ge -5$$

(b) Find an expression for $(gof)(x)$

Solution:

$$(gof)(x)=g(f(x))$$
$$=g(2(x+3)^2-5)$$
$$=\pm\sqrt{\frac{(2(x+3)^2-5)+5}{2}}$$
$$=-\sqrt{(x+3)^2} \qquad \{x\le -3 \Rightarrow x-3\le 0\}$$
$$=-(x+3)$$

Question 13

The function f and g are defined by:

$f : x \to x^3; \; x \in R$

$g : x \to 2-3x; x \in R$

(a) $(fog)(x)$

Solution:

$$(fog)(x) = f(g(x))$$
$$= f(2-3x)$$
$$= (2-3x)^3; \; x \in R$$

(b) $(fog)^{-1}(x)$

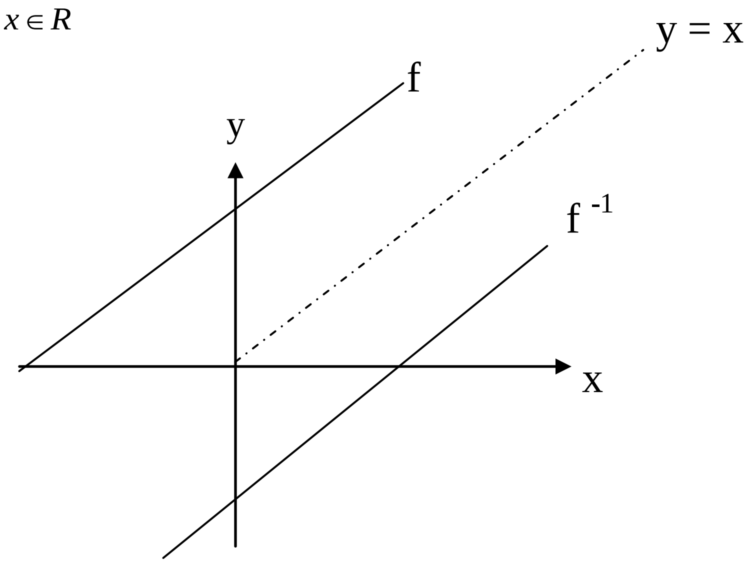
Solution:

$$(fog)(x) = (2-3x)^3$$
$$y = (2-3x)^3$$
$$2-3x = y^{1/3}$$
$$-3x = y^{1/3} - 2$$
$$x = \frac{y^{1/3} - 2}{-3}$$
$$= \frac{2 - y^{1/3}}{3}$$
$$(fog)^{-1}(y) = \frac{2 - y^{1/3}}{3}$$
$$(fog)^{-1}(x) = \frac{2 - x^{1/3}}{3}, \quad x \in R$$

Question 14
Sketch the graph of each of the following functions and its inverse.

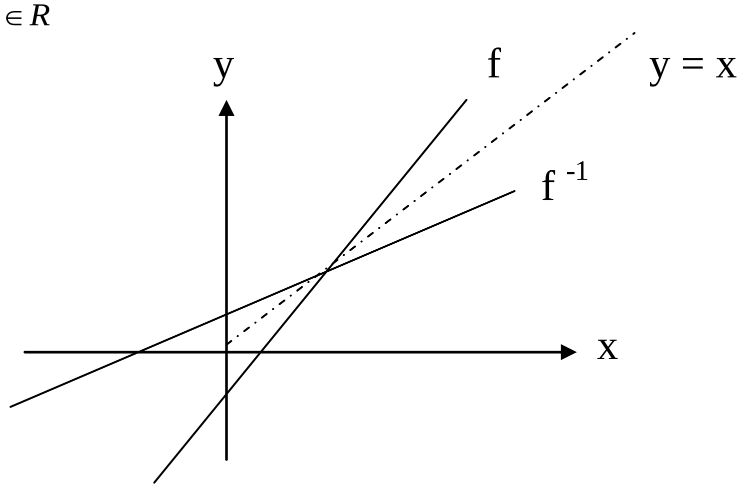
(a) $f(x) = x + 3; \; x \in R$

Solution:



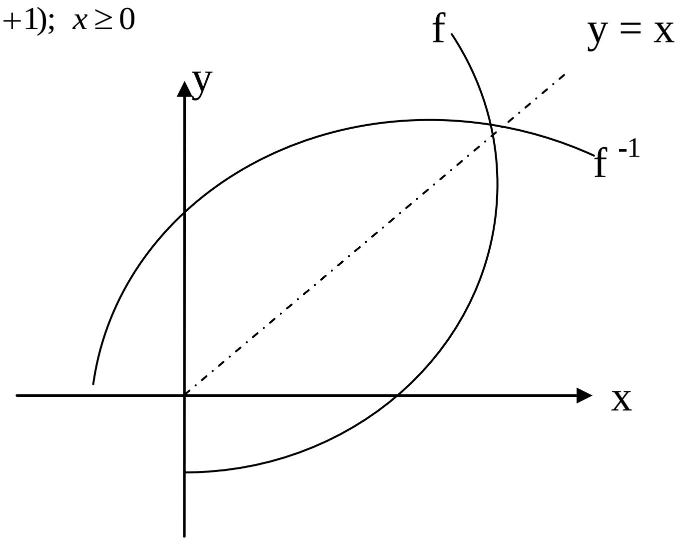
(b) $f(x) = 2x - 1; \; x \in R$

Solution:



(c) $f(x) = (x-1)(x+1);\ x \geq 0$

Solution:



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(d) $f(x) = x^2 + 3x - 4; x > -\frac{3}{2}$

Solution: