5.2 Lagrange interpolating polynomials ....

The Lagrange interpolating polynomial is simply a reformulation of the Newton polynomial that avoids the computation of divided differences. It can be represented concisely as

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$= L_0(x)f(x_0) + L_1(x)f(x_1) + \ldots + L_n f(x_n)$$

where $L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$

5.2.1 Linear interpolation

The linear version ($n = 1$) is

$$f_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Example

Given that (2,5) and (3,7). Use a first order Lagrange interpolating polynomial to evaluate *f*(2.5)

$$x_0 = 2, f(x_0) = 5; x_1 = 3, f(x_1) = 7$$

$$L_0(x) = \frac{x-3}{-1};\ L_1(x) = \frac{x-2}{1}$$

$$\therefore f_1(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$$

$$f_1(2.5) = L_0(2.5) f(x_0) + L_1(2.5) f(x_1)$$

$$= 0.5(5) + 0.5(7) = 6$$

5.2.2 Quadratic interpolation

The second order version ($n = 2$) is

$$f_2(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$$

Where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Example:

Given that (1,2), (2,5) and (3,7). Use a second order Lagrange interpolating to evaluate *f*(2.5)

$$x_0 = 1, f(x_0) = 2; x_1 = 2, f(x_1) = 5; x_2 = 3, f(x_2) = 7$$

$$L_0(x) = \frac{(x-2)(x-3)}{2}; L_1(x) = \frac{(x-1)(x-3)}{-1}; L_2(x) = \frac{(x-1)(x-2)}{2}$$

$$\therefore f_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$f_2(2.5) = L_0(2.5)f(x_0) + L_1(2.5)f(x_1) + L_2(2.5)f(x_2)$$

$$= -0.125(2) + 0.75(5) + 0.375(7) = 6.125$$

5.1.1 Linear interpolation

The simplest form of interpolation is to connect **2** data points with a straight line. This technique, called **linear interpolation**, and the linear-interpolation formula is given as

$$f_1(x) = b_0 + b_1(x - x_0) \qquad \text{or}$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

The notation $f_1(x)$ designates that this is a first-order interpolating polynomials. The term b_1 or $\frac{f(x_1)-f(x_0)}{x_1 - x_0}$ representing the slope of the line connecting the points.

The generation of the first divided difference outlined in table below.

X	f(x)	First divided difference $f[x_1, x_0]$
x_0	$f(x_0)=b_0$	
		$f[x_1,x_0]=\frac{f(x_1)-f(x_0)}{x_1 - x_0}=b_1$
x_1	$f(x_1)$	

Table 5.1

Example:

For the function cos (-*x*), given that

x	0.5	1.0	1.5	2.0
f(*x*)	0.8776	0.5403	0.0707	-0.4161

By using linear interpolation with *h* = 1.0 and *h* = 0.5 estimate cos(-1.0287)

when *h* = 1.0,

x	*f*(*x*)	First divided difference
1.0	$0.5403 = b_0$	
		$f[x_1, x_0] = \frac{-0.4161 - 0.5403}{2.0 - 1.0} = -0.9564 = b_1$
2.0	-0.4161	

General form of linear interpolation is

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$\therefore f_1(1.0287) = 0.5403 + (-0.9564)(1.0287 - 1.0)$$

$$= 0.5129$$

$$= 0.5129$$

when h = 0.5,

x	$f(x)$	First divided difference
1.0	$0.5403 = b_0$	
		$f[x_1, x_0] = \dfrac{0.0707 - 0.5403}{1.5 - 1.0} = -0.9392 = b_1$
1.5	0.0707	

General form of linear interpolation is

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$\therefore f_1(1.0287) = 0.5403 + (-0.9392)(1.0287 - 1.0)$$

$$= 0.5133$$

- Note that the true value of cos(-1.0287) is 0.515932898, we find that the smaller the interval between data points, the better the approximation.

5.1.2 Quadratic interpolation

The error could occur in linear interpolation resulted from our approximating a curve with straight line. Consequently, a strategy for improving the estimate is to introduce some curvature into the line connecting the points. If **3** data points are available, this can be accomplished with a **quadratic polynomial** (also called a second-order polynomial or parabola).

A particular convenient form for this purpose is

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

or

$$f_2(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

where the generation of the second divided difference(-dd) outlined in table below.

x	f(x)	First divided difference	Second divided difference
x_0	$f(x_0) = b_0$		
		$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = b_1$	
x_1	$f(x_1)$		$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = b_2$
		$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	
x_2	$f(x_2)$		

Table 5.2

Example:

For the function e^{-x}, given that

x	0.91	0.92	0.93	0.94
$f(x)$	0.4025	0.3985	0.3946	0.3906

By using quadratic interpolation, estimate $e^{-0.9321}$

x	$f(x)$	First divided difference	Second divided difference
0.92	$0.3985 = b_0$		
		$\frac{0.3946 - 0.3985}{0.93 - 0.92} = -0.39 = b_1$	
0.93	0.3946		$\frac{-0.4 - (-0.39)}{0.94 - 0.92} = -0.5 = b_2$
		$\frac{0.3906 - 0.3946}{0.94 - 0.93} = -0.4$	
0.94	0.3906		

General form of quadratic interpolation is

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$\begin{aligned}\therefore f_2(0.9321) &= 0.3985 + (-0.39)(0.9321 - 0.92) \\ &\quad + (-0.5)(0.9321 - 0.92)(0.9321 - 0.93) \\ &= 0.4217\end{aligned}$$

5.1.3 Cubic interpolation

For third-order interpolating polynomial, with n = 3, we yield

$$f_3(x) = f(x_0) + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

or

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

where the generation of the third divided difference(-dd) outlined in table below.

x	f(x)	1st -dd	2nd-dd	3rd-dd
x_0	$f(x_0) = b_0$			
		$f[x_1, x_0] = b_1$		
x_1	$f(x_1)$		$f[x_2, x_1, x_0] = b_2$	
		$f[x_2, x_1]$		$f[x_3, x_2, x_1, x_0] = \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3 - x_0} = b_3$
x_2	$f(x_2)$		$f[x_3, x_2, x_1]$	
		$f[x_3, x_2]$		
x_3	$f(x_3)$			

Table 5.3

Example:

For the function e^{-x}, given that

x	0.91	0.92	0.93	0.94
$f(x)$	0.4025	0.3985	0.3946	0.3906

By using cubic interpolation, estimate $e^{-0.9321}$

x	$f(x)$	1st -dd	2nd-dd	3rd-dd
0.91	$0.4025 = b_0$			
		$-0.4 = b_1$		
0.92	0.3985		$0.5 = b_2$	
		-0.39		$f[x_3, x_2, x_1, x_0] = \frac{-0.5-0.5}{0.94-0.91} = -33.33 = b_3$
0.93	0.3946		-0.5	
		-0.4		
0.94	0.3906			

General form of cubic interpolation is

$$f_3(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

$$f_3(0.9321) = 0.4025 + (-0.4)(0.9321 - 0.91) + 0.5(0.9321 - 0.91)(0.9321 - 0.92)$$
$$+ (-33.33)(0.9321 - 0.91)(0.9321 - 0.92)(0.9321 - 0.92)$$
$$= 0.3937$$