CHAPTER SEVEN

POLYNOMIALS

Introduction

In this chapter we will discuss about the algebraic operations on polynomial which are addition, subtraction, multiplication and this topic will introduce to long and synthetic division. Furthermore we will learn about remainder theorem, factor theorem and root and zeros of polynomial. Finally partial fractions will be discussed which involved linear factor in denominator, quadratic factor in denominator and repeated factor in denominator.

Objectives

After completing these tutorials, students should be able to:

- Find the quotient and remainder of the given function.
- Find the remainder when the functions are divided by the linear factors indicated.
- Find the third degree polynomial by using Cramer's Rule.
- Determine whether the given linear functions are factors of the given polynomials.
- Express the given proper algebraic fractions as partial fractions.
- Express the given improper algebraic fractions as partial fractions.

Question 1

Find the quotient and remainder of the following function by

i. Long division
ii. Synthetic division

Write the answer in the form $P(x) = Q(x)D(x) + R(x)$

(a) $\dfrac{x^3 - 27}{x - 3}$

Solution:

i. Long division

$$
\begin{array}{r}
x^2 + 3x + 9 \\
x - 3 \,\overline{)\, x^3 + 0x^2 + 0x - 27} \\
\underline{-(x^3 - 3x^2)} \\
3x^2 + 0x - 27 \\
\underline{-(3x^2 - 9x)} \\
9x - 27 \\
\underline{-(9x - 27)} \\
0
\end{array}
$$

ii. Synthetic division

$$
\begin{array}{r|rrr|r}
3 & 1 & 0 & 0 & -27 \\
(+) & & 3 & 9 & 27 \\
\hline
 & 1 & 3 & 9 & 0
\end{array}
$$

$\therefore P(x) = (x^2 + 3x + 9)(x - 3)$

(b) $\dfrac{-4x^3 + 2x^2 - x + 1}{x + 2}$

Solution:

iii. Long division

$$
\begin{array}{r}
-4x^2 + 10x - 21 \\
x + 2 \,\overline{)\, -4x^3 + 2x^2 - x + 1} \\
\underline{-(-4x^3 - 8x^2)} \\
10x^2 - x + 1 \\
\underline{-(10x^2 + 20x)} \\
-21x + 1 \\
\underline{-(-21x - 42)} \\
43
\end{array}
$$

iv. Synthetic division

–2	–4	2	–1	1
(+)		8	–20	42
	–4	10	–21	43

$Q(x) = -4x^2 + 10x - 21$

$R(x) = 43; D(x) = x + 2$

$\therefore P(x) = (-4x^2 + 10x - 21)(x + 2) + 43$

Question 2

Find the remainder when the following functions are divided by the linear factors indicated.

(a) $x^3 - 2x + 4; x - 1$

Solution:

$\therefore a = 1$

$P(1) = (1)^3 - 2(1) + 4 = 3$

(b) $x^4 - 3x^3 + 5x; 2x - 1$

Solution:

$\therefore a = \frac{1}{2}$

$P\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right) = \frac{35}{16}$

Question 3

When $P(x)=2x^3+kx^2+k^2x+13$ is divided by $(x-k)$, the remainder is 269. Find the value of *k*.

Solution:

$P(x)=2x^3+kx^2+k^2x+13$

$D(x)=(x-k)\rightarrow a=k$

$R(x)=269$

$2(k)^3+k(k)^2+k^2(k)+13=269$

$2k^3+k^3+k^3+13=269$

$4k^3=256$

$k^3=64$

$k=\sqrt[3]{64}=4$

Question 4

The remainder of $P(x)=x^3-2x+r$ when it is divided by $(x-1)$, is the same as the remainder when $Q(x)=2x^3+x-r$ is divided $(2x+1)$, find the value of r.

Solution:

$P(x)=x^3-2x+r; D_1(x)=(x-1)\rightarrow a_1=1$

$Q(x)=2x^3+x-r; D_2(x)=(2x+1)\rightarrow a_2=-\frac{1}{2}$

So,

$$x^3-2x+r=2x^3+x-r$$

$$(1)^3-2(1)+r=2\left(\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)-r$$

$$1-2+r=-\frac{2}{8}-\frac{1}{2}-r$$

$$2r=-\frac{6}{8}+1$$

$$r=\frac{1}{8}$$

Question 5

Find the third degree polynomial $P(x)$ if $P(1) = P(-2) = 0, P(-1) = 4$ and $P(2) = 28$

Solution:

$P(1) = 0; P(-1) = 4; P(-2) = 0; P(2) = 28.$

Third degree polynomial ($ax^3 + bx^2 + cx + d$)

So

$$a(1)^3 + b(1)^2 + c(1) + d = 0$$
$$a(-2)^3 + b(-2)^2 + c(-2) + d = 0$$
$$a(-1)^3 + b(-1)^2 + c(-1) + d = 4$$
$$a(2)^3 + b(2)^2 + c(2) + d = 28$$

then

$$a + b + c + d = 0$$
$$-8a \_ 4b - 2c + d = 0$$
$$-a + b - c + d = 4$$
$$8a + 4b + 2c + d = 28$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -8 & 4 & -2 & 1 \\ -1 & 1 & -1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 28 \end{bmatrix}$$
$$A \qquad x = B$$

Using the Cramer's Rule

$|A| = 72$

$$a = \frac{216}{72} = 3; b = \frac{288}{72} = 4; c = \frac{-360}{72} = -5; d = \frac{-144}{72} = -2$$

$\therefore 3x^3 + 4x^2 - 5x - 2$

Question 6

Determine whether the following linear functions are factors of the given polynomials.

(a) $2x^2 + 3x - 4; x + 1$

Solution:

$a = -1$

$P(-1) = 2(-1)^2 + 3(-1) - 4 = -5$

(not a factor)

(b) $2x^4 - x^3 - 6x^2 + 5x - 1; 2x - 1$

Solution:

$a = \frac{1}{2}$

$P\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4 - \left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right) = \frac{1-1-12+20-8}{8} = 0$

(a factor)

Question 7

Given $(x-1)$ and $(x+2)$ are factors of $P(x) = 2x^4 - 5x^3 + ax^2 + bx + 8$, find the values of a and b. Hence factorize $P(x)$ completely.

Solution:

$P(x) = 2x^4 - 5x^3 + ax^2 + bx + 8$

$(x-1) \Rightarrow a = 1$

$2(1)^4 - 5(1)^3 + a(1)^2 + b(1) + 8 = 0$

$a + b = -5$ ……………….. (1)

$(x+2) \Rightarrow a = -2$

$2(-2)^4 - 5(-2)^3 + a(-2)^2 + b(-2) + 8 = 0$

$2a - b = -40$ ………………………(2)

(2) + (1)

$3a = -45$

$a = -15$

$a = -15$ in (1)

$-15 + b = -5$

$b = 10$

$\therefore a = -15; b = 10$

$\Rightarrow P(x) = 2x^4 - 5x^3 - 15x^2 + 10x + 8$

By synthetic division

1	2	–5	–15	10	8
(+)		2	–3	–18	–8
–2	2	–3	–18	–8	0
(+)		–4	14	8	
	2	–7	–4	0	

$P(x) = (2x^2 - 7x - 4)(x-1)(x+2) = (2x+1)(x-4)(x-1)(x+2)$

Question 8

Express the following as partial fractions.

(a) $\dfrac{5x+29}{(x-4)(x+3)}$

Solution:

$$\frac{5x+29}{(x-4)(x+3)}=\frac{A}{(x-4)}+\frac{B}{(x+3)}$$

$$\frac{5x+29}{(x-4)(x+3)}=\frac{A(x+3)+B(x-4)}{(x-4)(x+3)}$$

$5x+29=A(x+3)+B(x-4)$

when $x=-3$

$5(-3)+29=A(-3+3)+B(-3-4)$

$7B=-14$

$B=-2$

when $x=4$

$5(4)+29=A(4+3)+B(4-4)$

$7A=49$

$A=7$

So, $\dfrac{5x+29}{(x-4)(x+3)}=\dfrac{7}{(x-4)}-\dfrac{2}{(x+3)}$

(b) $\dfrac{18x-9}{(x+2)(x-1)^2}$

Solution:

$$\frac{18x-9}{(x+2)(x-1)^2}=\frac{A}{(x+2)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

$$\frac{18x-9}{(x+2)(x-1)^2}=\frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}$$

$\therefore 18x-9=A(x-1)^2+B(x+2)(x-1)+C(x+2)$

when $x = 1$

$18(1)-9=A(1-1)^2+B(1+2)(1-1)+C(1+2)$

$9=3C$

$C=3$

when $x=-2$

$18(-2)-9=A(-2-1)^2+B(-2+2)(-2-1)+C(-2+2)$

$45=9A$

$A=-5$

when $x=0$

$18(0)-9=A(0-1)^2+B(0+2)(0-1)+C(0+2)$

$2B=10$

$B=5$

So, $\dfrac{18x-9}{(x+2)(x-1)^2}=\dfrac{-5}{(x+2)}+\dfrac{5}{(x-1)}+\dfrac{3}{(x-1)^2}$

(c) $\dfrac{x^2+4x+3}{(x^2+2x+2)^2}$

Solution:

$\dfrac{x^2+4x+3}{(x^2+2x+2)^2}=\dfrac{Ax+B}{(x^2+2x+2)}+\dfrac{Cx+D}{(x^2+2x+2)^2}$

$x^2+4x+3=(Ax+B)(x^2+2x+2)^2+(Cx+D)$

$x^2+4x+3=Ax^3+(2A+B)x^2+(2A+2B+C)x+(2B+D)$

Compare the coefficient

$A=0;$

$2A+B=1$

$B=1;$

$2A+2B+C=4$

$C=2;$

$2B+D=3$

$D=1;$

$\therefore \dfrac{x^2+4x+3}{(x^2+2x+2)^2}=\dfrac{1}{(x^2+2x+2)}+\dfrac{2x+1}{(x^2+2x+2)^2}$

Question 9

Express the following improper algebraic fractions as partial fractions.

(a) $$\frac{2x^2+3x+2}{x^2+3x+2}$$

Solution:

$$\frac{2x^2+3x+2}{x^2+3x+2}=\frac{2x^2+3x+2}{(x+1)(x+2)}$$

by long division

$$\begin{array}{r|l} & 2 \\ x^2+3x+2 & 2x^2+3x+2 \\ & 2x^2+6x+4 \\ & \quad -3x-2 \end{array}$$

$$\therefore \frac{2x^2+3x+2}{x^2+3x+2}=2+\frac{(-3x-2)}{x^2+3x+2}$$

$$\Rightarrow \frac{-3x-2}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

$-3x-2=A(x+2)+B(x+1)$

when $x=-2$

$B=-4$

when $x=-1$

$A=1$

so, $$\frac{2x^2+3x+2}{x^2+3x+2}=2+\frac{1}{(x+1)}-\frac{4}{(x+2)}$$

(b) $$\frac{x^3-1}{(x+1)(x-2)}$$

Solution:

$$\frac{x^3-1}{(x+1)(x-2)}=\frac{x^3-1}{x^2-x-2}$$

$$\begin{array}{r|l} & x+1 \\ x^2-x-2 & x^3+0x^2+0x-1 \\ & -(x^3-x^2-2x) \\ \hline & \quad x^2+2x-1 \\ & \quad -(x^2-x-2) \\ \hline & \qquad 3x+1 \end{array}$$

$$\frac{x^3-1}{(x+1)(x-2)} = (x+1) + \frac{3x+1}{(x+1)(x-2)}$$

$$\Rightarrow \frac{3x+1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

$$3x+1 = A(x-2) + B(x+1)$$

when $x = 2 \Rightarrow B = \frac{7}{3}$

when $x = -1 \Rightarrow A = \frac{2}{3}$

so, $\frac{x^3-1}{(x+1)(x-2)} = (x+1) + \frac{2}{3(x+1)} + \frac{7}{3(x-2)}$

(c) $\frac{x^3-4x}{(x-2)(x+1)^2}$

Solution:

$$\frac{x^3-4x}{(x-2)(x+1)^2} = \frac{x^3-4x}{x^3-3x-2}$$

By long division

$$\begin{array}{r|l} & 1 \\ x^3-3x-2 & x^3-4x \\ & -(x^3-3x-2) \\ \hline & -x+2 \end{array}$$

$$\frac{x^3-4x}{(x-2)(x+1)^2} = 1 + \frac{(-x+2)}{(x-2)(x+1)^2}$$

$$\Rightarrow \frac{(-x+2)}{(x-2)(x+1)^2} = \frac{A}{(x-2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$-x+2 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)$$

when $x = -1; C = -1$

when $x = 2; A = 0$

when $x = 0; B = 0$

so, $\frac{x^3-4x}{(x-2)(x+1)^2} = 1 - \frac{1}{(x+1)^2}$