9.1 The power method ....

The power method is an iterative approach that can be employed to determine the largest eigen value and corresponding eigen vector.

If matrix **A** has n eigen vectors { $\mathbf{v}_1$, $\mathbf{v}_2$, … ,$\mathbf{v}_n$} which are linear independent, hence any nonzero vector $\mathbf{v}^{(0)}$ can be represented as

$$\mathbf{v}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

$$= \sum_{i=1}^{n} c_i\mathbf{v}_i$$

where c_i is a constant, $i = 1,2, \ldots, n$

So,

$$\mathbf{A}\mathbf{v}^{(0)} = \mathbf{A}\left(\sum_i^n c_i\mathbf{v}_i\right) = \sum_i^n c_i\lambda_i\mathbf{v}_i$$

$$\mathbf{v}^{(1)} = \frac{1}{m_1}\mathbf{A}\mathbf{v}^{(0)} = \frac{1}{m}\sum_i^n c_i\lambda_i\mathbf{v}_i$$

where m_1 is an entry of $\mathbf{A}\mathbf{v}^{(0)}$ which has the highest absolute value.

In general, the iteration formula of power method can be written as

$$\mathbf{v}^{(k+1)} = \frac{1}{m_{k+1}} \mathbf{A}\mathbf{v}^{(k)}, \; k = 0, 1, 2, \ldots$$

where m_{k+1} is an entry of $\mathbf{A}\mathbf{v}^{(k)}$ which has the highest absolute value.

When $\left\| \mathbf{v}^{(k)} - \mathbf{v}^{(k-1)} \right\|_\infty = \max_{1 \le i \le n} \left| \mathbf{v}_i^{(k)} - \mathbf{v}_i^{(k-1)} \right| < \varepsilon$

The computation is terminated for a prespecified stopping criterion ε. The result is converging on a **highest eigen value** $\lambda \approx \boldsymbol{m_{k+1}}$ with a corresponding **eigen vector** $\mathbf{v} \approx \mathbf{v}^{(k)}$.

Example:

Employ the power method to determine the highest eigen value and corresponding eigen vector for

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

accurate to within $\varepsilon = 0.005$.

Assume $v^{(0)} = (1,1,1)^T$

So $$\mathbf{Av}^{(0)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; \qquad m_1 = 1$$

$$\mathbf{v}^{(1)} = \frac{1}{m_1}\left(\mathbf{Av}^{(0)}\right) = \frac{1}{1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left\|\mathbf{v}^1 - \mathbf{v}^0\right\|_\infty = 1$$

$$\mathbf{Av}^{(1)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}; \qquad m_2 = 2$$

$$\mathbf{v}^{(2)} = \frac{1}{m_2}\left(\mathbf{Av}^{(1)}\right) = \frac{1}{2}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\left\|\mathbf{v}^2 - \mathbf{v}^1\right\|_\infty = 1$$

$$\mathbf{Av}^{(2)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix}; \qquad m_3 = -4$$

$$\mathbf{v}^{(3)} = \frac{1}{m_3}\left(\mathbf{Av}^{(2)}\right) = \frac{1}{-4}\begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -0.75 \\ 1 \\ -0.75 \end{pmatrix}$$

$$\left\|\mathbf{v}^3 - \mathbf{v}^2\right\|_\infty = 2$$

$$\mathbf{Av}^{(3)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -0.75 \\ 1 \\ -0.75 \end{pmatrix} = \begin{pmatrix} -2.5 \\ 3.5 \\ -2.5 \end{pmatrix}; \qquad m_4 = 3.5$$

$$\mathbf{v}^{(4)} = \frac{1}{m_4}\left(\mathbf{Av}^{(3)}\right) = \frac{1}{3.5}\begin{pmatrix} -2.5 \\ 3.5 \\ -2.5 \end{pmatrix} = \begin{pmatrix} -0.714 \\ 1 \\ -0.714 \end{pmatrix}$$

$$\left\|\mathbf{v}^4 - \mathbf{v}^3\right\|_\infty = 0.036$$

$$\mathbf{Av}^{(4)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -0.714 \\ 1 \\ -0.714 \end{pmatrix} = \begin{pmatrix} -2.428 \\ 3.428 \\ -2.428 \end{pmatrix}; \qquad m_5 = 3.428$$

$$\mathbf{v}^{(5)} = \frac{1}{m_5}\left(\mathbf{Av}^{(4)}\right) = \frac{1}{3.428}\begin{pmatrix} -2.428 \\ 3.428 \\ -2.428 \end{pmatrix} = \begin{pmatrix} -0.708 \\ 1 \\ -0.708 \end{pmatrix}$$

$$\left\|\mathbf{v}^5 - \mathbf{v}^4\right\|_\infty = 0.006$$

$$\mathbf{Av}^{(5)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -0.708 \\ 1 \\ -0.708 \end{pmatrix} = \begin{pmatrix} -2.416 \\ 3.416 \\ -2.416 \end{pmatrix}; \qquad m_6 = 3.416$$

$$\mathbf{v}^{(6)} = \frac{1}{m_6}\left(\mathbf{Av}^{(5)}\right) = \frac{1}{3.416}\begin{pmatrix} -2.416 \\ 3.416 \\ -2.416 \end{pmatrix} = \begin{pmatrix} -0.707 \\ 1 \\ -0.707 \end{pmatrix}$$

$$\left\|\mathbf{v}^6 - \mathbf{v}^5\right\|_\infty = 0.001$$

$$\mathbf{Av}^{(6)} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} -0.707 \\ 1 \\ -0.707 \end{pmatrix} = \begin{pmatrix} -2.414 \\ 3.414 \\ -2.414 \end{pmatrix}; \qquad m_7 = 3.414$$

$\therefore$ Eigen value, $\lambda = m_7 = 3.414$

Eigen vector, $\mathbf{v} = \mathbf{v}^{(6)} = \begin{pmatrix} -0.707 \\ 1 \\ -0.707 \end{pmatrix}$